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About the unification type of $\mathbf{K} + \Box\Box\bot$

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Abstract

The unification problem in a propositional logic is to determine, given a formula φ, whether there exists a substitution σ such that $\sigma(\varphi)$ is in that logic. In that case, σ is a unifier of φ. When a unifiable formula has minimal complete sets of unifiers, it is either infinitary, finitary, or unitary, depending on the cardinality of its minimal complete sets of unifiers. Otherwise, it is nullary. In this paper, we prove that in modal logic $\mathbf{K} + \Box\Box\bot$, unifiable formulas are either finitary, or unitary.

Keywords Propositional modal logics · Locally tabular modal logics · Unification problem · Unification types

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1 Introduction

The unification problem in a propositional logic is to determine, given a formula φ, whether there exists a substitution σ such that $\sigma(\varphi)$ is in that logic. In that case, σ is a unifier of φ. We say that a set of unifiers of a unifiable formula φ is complete if for all unifiers σ of φ, there exists a unifier τ of φ in that set such that τ is more general than σ.[1] Now, an important question is to determine whether a given unifiable formula has minimal complete sets of unifiers [2]. When such sets exist, they all have the same cardinality. In that case, a unifiable formula is either infinitary, or finitary, or unitary, depending whether its complete sets of unifiers are either infinite, or finite, or with cardinality 1. Otherwise, the formula is nullary.

[1] A substitution σ is more general than a substitution τ in a propositional logic if there exists a substitution υ such that for all variables x, $\upsilon(\sigma(x)) \leftrightarrow \tau(x)$ is in that logic.

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The importance of the unification problem lies in its connection with the admissibility problem. In a consistent propositional logic **L**, unification is reducible to non-admissibility, seeing that the unifiability in **L** of a formula φ is equivalent to the non-admissibility in **L** of the inference rule $\frac{\varphi}{\bot}$. As observed by Ghilardi [17], when **L** has a decidable membership problem and **L** is either unitary, or finitary, algorithms for computing minimal complete sets of unifiers in **L** can be used as a key component of algorithms for solving the admissibility problem in **L**, seeing that the admissibility in **L** of an inference rule $\frac{\varphi_1,\ldots,\varphi_p}{\psi}$ is equivalent to the inclusion in **L** of the set $\{\sigma(\psi) : \sigma\in\Sigma\}$, where Σ is an arbitrary minimal complete set of unifiers of $\varphi_1 \wedge \ldots \wedge \varphi_p$ in **L**.

Within the context of the unification problem in a propositional logic, we distinguish between elementary unification and unification with constants. In unification with constants, some variables (called constants) are never replaced by formulas when one applies a substitution whereas in elementary unification, all variables are likely to be replaced. About the unification type of modal logics, it is known that **KT**, **KD** and **KB** are nullary [4, 5, 7], **KD**45 and **K**45 are unitary [6, 10, 19, 22], $\mathbf{Alt}_1 + \Box^d\bot$ (the least modal logic containing $\mathbf{Alt}_1$ and $\Box^d\bot$) is unitary for each $d\geq 2$ [8], **S**5 and **S**4.3 are unitary [13–15], transitive modal logics like **K**4 and **S**4 are finitary [17, 21], **K** is nullary [23] and **K**4**D**1 is unitary [24], the type of **KT**, **KD** and **KB** having only been obtained within the context of unification with constants and the type of $\mathbf{Alt}_1 + \Box^d\bot$ having only been obtained within the context of elementary unification.[2] About the unification type of $\mathbf{Alt}_1$ and its extensions, the line of reasoning determining in [5, 7] the unification type (nullary) of **KD** within the context of unification with constants can be adapted to $\mathbf{Alt}_1 + \Diamond\top$ whereas the line of reasoning determining in [23] the unification type (nullary) of **K** has been adapted to $\mathbf{Alt}_1$ [9]. In this paper, within the context of elementary unification, we prove that in $\mathbf{K} + \Box\Box\bot$ (the least modal logic containing $\Box\Box\bot$), unifiable formulas are either finitary, or unitary.[3]

2 A preliminary result

Let S be a finite set. We write $\|S\|$ for the cardinality of S. If S is nonempty then for all equivalence relations $\sim$ on S, for all $\alpha\in S$, $[\alpha]$ denotes the equivalence class of α modulo $\sim$ and for all $T\subseteq S$, $T/\sim$ denotes the quotient set of T modulo $\sim$. Our first result, Proposition 1, is used later in Section 6. Its proof is presented in an Appendix along with the proofs of most of the results asserted in this paper.

Proposition 1 *Let T be a finite set. If S is nonempty then for all equivalence relations $\sim$ on S, $\|S/\sim\|\leq\|T\|\leq\|S\|$ iff there exists a surjective function f from S to T such that for all $\alpha, \beta\in S$, if $f(\alpha)=f(\beta)$ then $\alpha\sim\beta$.*

[2] In this paper, all modal logics are normal. We follow the same conventions as in [11, 12, 25] for talking about them: **S**5 is the least modal logic containing the formulas usually denoted **T**, 4 and 5, **KD** is the least modal logic containing the formula usually denoted **D**, etc. In particular, $\mathbf{Alt}_1$ is the least modal logic containing $\Diamond x \to \Box x$ and **K**4**D**1 is the least modal logic containing **K**4 and $\Box(\Box x \to y) \vee \Box(\Box y \to x)$.

[3] From now on, "... iff ..." means "... if and only if ..." and "... not-iff ..." means "... if and only if not ...".

3 Syntax

Let **VAR** be a countably infinite set of *variables* (with typical members denoted x, y, etc). Let $(x_1, x_2, \ldots)$ be an enumeration of **VAR** without repetitions. Let $n\geq 1$. The set $\mathbf{FOR}_n$ of all *n-formulas* (with typical members denoted φ, ψ, etc) is inductively defined by:

- $\varphi, \psi ::= x_i \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$.

We adopt the standard rules for omission of the parentheses. The connectives $\top$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined by the usual abbreviations. We have also a connective $\Diamond$ which is defined by

- $\Diamond\varphi ::= \neg\Box\neg\varphi$.

For all $\varphi\in\mathbf{FOR}_n$, we respectively write "φ^0" and "φ^1" to mean "$\neg\varphi$" and "φ". An *n-substitution* is a couple (k, σ) where $k\geq 1$ and σ is a homomorphism from $\mathbf{FOR}_n$ to $\mathbf{FOR}_k$. Let $\mathbf{SUB}_n$ be the set of all n-substitutions. From now on,

we write "$\mathbf{L}_2$" to mean "$\mathbf{K}+\Box\Box\bot$".

The standard axiomatization of $\mathbf{L}_2$ consists of the following axioms and rules of proof:

- all propositional tautologies,
- $\Box(x_i \rightarrow x_j) \rightarrow (\Box x_i \rightarrow \Box x_j)$,
- $\Box\Box\bot$,
- modus ponens,
- uniform substitution,
- generalization: given φ, prove $\Box\varphi$.

As is well-known, $\mathbf{L}_2$ is the modal logic of directed graphs where there is no path of length 3. The generated subgraphs of such directed graphs are therefore tree-like Kripke frames of depth at most 1. They constitute the semantic basis of $\mathbf{L}_2$ — tree-like Kripke models of depth at most 1 — developed in Section 4. In the meantime, let us consider the equivalence relation $\equiv_n$ on $\mathbf{FOR}_n$ defined by:

- $\varphi\equiv_n\psi$ iff $\varphi \leftrightarrow \psi\in\mathbf{L}_2$.

Proposition 2 $\equiv_n$ *possesses finitely many equivalence classes.*

In other words, $\mathbf{L}_2$ is *locally tabular*. Locally tabular modal logics possess interesting properties. In particular, in terms of decidability [26–28]. See also [12, Chapter 12] and [16]. The equivalence relation $\simeq_n$ on $\mathbf{SUB}_n$ is defined by:

- $(k, \sigma)\simeq_n(l, \tau)$ iff for all $i\in\{1, \ldots, n\}$, $\sigma(x_i) \leftrightarrow \tau(x_i)\in\mathbf{L}_2$.

The preorder $\preccurlyeq_n$ on $\mathbf{SUB}_n$ is defined by:

- $(k, \sigma)\preccurlyeq_n(l, \tau)$ iff there exists a k-substitution (m, υ) such that for all $i\in\{1, \ldots, n\}$, $\upsilon(\sigma(x_i)) \leftrightarrow \tau(x_i)\in\mathbf{L}_2$.

4 Semantics

Let $n\geq 1$. An *n-tuple of bits* (denoted α, β, etc) is a function from $\{1, \ldots, n\}$ to $\{0, 1\}$. Such function should be understood as a propositional valuation of the variables $x_1, \ldots, x_n$: for all $i\in\{1, \ldots, n\}$, if $\alpha_i=0$ then it is interpreted to mean "x_i is false" else it is interpreted to mean "x_i is true". Let $\mathbf{BIT}_n$ be the set of all n-tuples of bits. An *n-model* is a structure of the form (α, S) where $\alpha \in \mathbf{BIT}_n$ and $S \subseteq \mathbf{BIT}_n$. Such structure should be understood as a tree-like Kripke model of depth at most 1.[4] Let $\mathbf{MOD}_n$ be the set of all n-models. We say that an n-model (α, S) is *degenerated* if $S = \emptyset$. Let $\mathbf{MOD}_n^{\mathrm{deg}}$ be the set of all degenerated n-models. Notice that $\|\mathbf{MOD}_n^{\mathrm{deg}}\| = 2^n$. Notice also that for all sets S of n-tuples of bits, $S \times \{\emptyset\}$ is a set of degenerated n-models. The binary relation $\models_n$ of *n-satisfiability* between $\mathbf{MOD}_n$ and $\mathbf{FOR}_n$ is defined as expected:

- $(\alpha, S) \models_n x_i$ iff $\alpha_i = 1$,
- $(\alpha, S) \not\models_n \bot$,
- $(\alpha, S) \models_n \neg\varphi$ iff $(\alpha, S) \not\models_n \varphi$,
- $(\alpha, S) \models_n \varphi \vee \psi$ iff either $(\alpha, S) \models_n \varphi$, or $(\alpha, S) \models_n \psi$,
- $(\alpha, S) \models_n \Box\varphi$ iff for all $\beta \in S$, $(\beta, \emptyset) \models_n \varphi$.

As a result,

- $(\alpha, S) \models_n \Diamond\varphi$ iff there exists $\beta \in S$ such that $(\beta, \emptyset) \models_n \varphi$.

Obviously, for all $\alpha \in \mathbf{BIT}_n$ and for all $\varphi \in \mathbf{FOR}_n$, $(\alpha, \emptyset) \models_n \Box\varphi$ and $(\alpha, \emptyset) \not\models_n \Diamond\varphi$. We say that $\varphi \in \mathbf{FOR}_n$ is *n-valid* if for all $(\alpha, S) \in \mathbf{MOD}_n$, $(\alpha, S) \models_n \varphi$. The soundness and the completeness of $\mathbf{L}_2$ with respect to the concept of validity is well-known. This is what Proposition 3 is about.

Proposition 3 *For all $\varphi \in \mathbf{FOR}_n$, $\varphi \in \mathbf{L}_2$ iff φ is n-valid.*

For all $\alpha \in \mathbf{BIT}_n$, the n-formula

- $\bar{x}^{\alpha} = \bigwedge\{x_i^{\alpha_i} : i \in \{1, \dots, n\}\}$

exactly characterizes the propositional valuation of the variables $x_1, \dots, x_n$ represented by α. This is what Proposition 4 is about.

Proposition 4 *For all $(\alpha, S) \in \mathbf{MOD}_n$ and for all $\beta \in \mathbf{BIT}_n$, $\alpha = \beta$ iff $(\alpha, S) \models_n \bar{x}^{\beta}$.*

For all $(\alpha, S) \in \mathbf{MOD}_n$, the n-formula

- $\mathbf{for}_n(\alpha, S) = \bar{x}^{\alpha} \wedge \Box \bigvee\{\bar{x}^{\gamma} : \gamma \in S\} \wedge \bigwedge\{\Diamond\bar{x}^{\gamma} : \gamma \in S\}$

exactly characterizes the tree-like Kripke model of depth at most 1 represented by (α, S). This is what Proposition 5 is about.

Proposition 5 *For all $(\alpha, S), (\beta, T) \in \mathbf{MOD}_n$, $(\alpha, S) = (\beta, T)$ iff $(\alpha, S) \models_n \mathbf{for}_n(\beta, T)$.*

As we know, an n-substitution (k, σ) is a homomorphism from $\mathbf{FOR}_n$ to $\mathbf{FOR}_k$. Taking into account the duality between n-formulas and n-models, Propositions 6 and 7 tell us how to associate a function $g_{(k,\sigma)}$ from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ to any n-substitution (k, σ).[5]

[4] Indeed, an n-model (α, S) should be seen as a Kripke model (W, R, V) with set W of possible worlds the set $\{(\alpha, 0)\} \cup \{(\beta, 1) \colon \beta \in S\}$, with accessibility relation R the binary relation $\{((\alpha, 0), (\beta, 1)) \colon \beta \in S\}$ on W and with valuation V the function from $\{x_1, \dots, x_n\}$ to $\mathcal{P}(W)$ such that for all $i \in \{1, \dots, n\}$, if $\alpha_i = 0$ then $V(x_i) = \{(\beta, 1) \colon \beta \in S$ and $\beta_i = 1\}$ else $V(x_i) = \{(\alpha, 0)\} \cup \{(\beta, 1) \colon \beta \in S$ and $\beta_i = 1\}$.

[5] Propositions 6 and 7 state standard results connecting substitutions and models. In particular, Proposition 6 is an immediate consequence of [17, Proposition 1.3].

Proposition 6 *Let* $(k,\sigma)\in\mathbf{SUB}_n$. *For all* $(\alpha,S)\in\mathbf{MOD}_k$, *there exists* $(\beta,T)\in\mathbf{MOD}_n$ *such that* $(\alpha,S)\models_k\sigma(\mathbf{for}_n(\beta,T))$.

Proposition 7 *Let* $(k,\sigma)\in\mathbf{SUB}_n$. *Let* $(\alpha,S)\in\mathbf{MOD}_k$. *For all* $(\beta,T),(\gamma,U)\in\mathbf{MOD}_n$, *if* $(\alpha,S)\models_k\sigma(\mathbf{for}_n(\beta,T))$ *and* $(\alpha,S)\models_k\sigma(\mathbf{for}_n(\gamma,U))$ *then* $(\beta,T)=(\gamma,U)$.

For all $(k,\sigma)\in\mathbf{SUB}_n$, let $g_{(k,\sigma)}$ be the function from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ such that for all $(\alpha,S)\in\mathbf{MOD}_k$,

- $g_{(k,\sigma)}(\alpha,S)$ is the $(\beta,T)\in\mathbf{MOD}_n$ such that $(\alpha,S)\models_k\sigma(\mathbf{for}_n(\beta,T))$.

For all $(k,\sigma)\in\mathbf{SUB}_n$, notice that by Propositions 6 and 7, $g_{(k,\sigma)}$ is well-defined. Moreover, for all $(k,\sigma)\in\mathbf{SUB}_n$, for all $(\alpha,S)\in\mathbf{MOD}_k$ and for all $(\beta,T)\in\mathbf{MOD}_n$, if $g_{(k,\sigma)}(\alpha,S)=(\beta,T)$ then

- for all $\gamma\in S$, there exists $\delta\in T$ such that $g_{(k,\sigma)}(\gamma,\emptyset)=(\delta,\emptyset)$,
- for all $\delta\in T$, there exists $\gamma\in S$ such that $g_{(k,\sigma)}(\gamma,\emptyset)=(\delta,\emptyset)$.

See Proposition 8 below. Obviously, the above conditions are very similar to the forward condition and backward condition of bounded morphisms usually considered in modal logic [11, Definition 2.10]. This motivates the following definition. For all $k\geq1$, a (k,n)-*morphism* is a function f from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ such that for all $(\alpha,S)\in\mathbf{MOD}_k$ and for all $(\beta,T)\in\mathbf{MOD}_n$, if $f(\alpha,S)=(\beta,T)$ then

- for all $\gamma\in S$, there exists $\delta\in T$ such that $f(\gamma,\emptyset)=(\delta,\emptyset)$,
- for all $\delta\in T$, there exists $\gamma\in S$ such that $f(\gamma,\emptyset)=(\delta,\emptyset)$.

Proposition 8 *For all* $(k,\sigma)\in\mathbf{SUB}_n$, $g_{(k,\sigma)}$ *is a* (k,n)*-morphism.*

However, the morphisms described here should not be mistaken for the bounded morphisms. In particular, in the above definition, there is no condition related to the propositional valuations of the variables. For all $(k,\sigma)\in\mathbf{SUB}_n$, the best we can say about the propositional valuations of the variables concerns $g_{(k,\sigma)}$ and is contained in the following result.

Proposition 9 *For all* $(k,\sigma)\in\mathbf{SUB}_n$ *and for all* $(\alpha,S),(\beta,T)\in\mathbf{MOD}_k$, *if* $g_{(k,\sigma)}(\alpha,S)=g_{(k,\sigma)}(\beta,T)$ *then for all* $i\in\{1,\ldots,n\}$, $(\alpha,S)\models_k\sigma(x_i)$ *iff* $(\beta,T)\models_k\sigma(x_i)$.

Nevertheless, it is not particularly surprising that we have the following results.

Proposition 10 *Let* $k\geq1$. *Let* f *be a* (k,n)*-morphism. Let* $(\beta,T)\in\mathbf{MOD}_k$ *and* $(\gamma,U)\in\mathbf{MOD}_n$. *If* $f(\beta,T)=(\gamma,U)$ *then the image by* f *of* $T\times\{\emptyset\}$ *is equal to* $U\times\{\emptyset\}$. *Moreover,* $T=\emptyset$ *iff* $U=\emptyset$.

Proposition 11 *Let* $k\geq1$. *Let* f *be a* (k,n)*-morphism. Let* $(\beta,T)\in\mathbf{MOD}_k$ *and* $(\gamma,U)\in\mathbf{MOD}_n$. *If the following conditions hold then* $f(\beta,T)=(\gamma,U)$:

- $f(\beta,T)\models_n\bar{x}^\gamma$,
- *for all* $\delta\in T$, *there exists* $\epsilon\in U$ *such that* $f(\delta,\emptyset)=(\epsilon,\emptyset)$,
- *for all* $\epsilon\in U$, *there exists* $\delta\in T$ *such that* $f(\delta,\emptyset)=(\epsilon,\emptyset)$.

5 Unification

Let $n \geq 1$. An *n-unifier* of $\varphi \in \mathbf{FOR}_n$ is an n-substitution (k, σ) such that $\sigma(\varphi) \in \mathbf{L}_2$. We say that $\varphi \in \mathbf{FOR}_n$ is *n-unifiable* if there exists an n-unifier of φ. We say that a set Σ of n-unifiers of an n-unifiable $\varphi \in \mathbf{FOR}_n$ is *n-complete* if for all n-unifiers (k, σ) of φ, there exists $(l, \tau) \in \Sigma$ such that $(l, \tau) \preccurlyeq_n (k, \sigma)$. We say that an n-complete set Σ of n-unifiers of an n-unifiable $\varphi \in \mathbf{FOR}_n$ is *minimal* if for all n-complete sets Δ of n-unifiers of φ, if $\Delta \subseteq \Sigma$ then $\Delta = \Sigma$. As is well-known, for all $\varphi \in \mathbf{FOR}_n$, if φ is n-unifiable then for all minimal n-complete sets Σ, Δ of n-unifiers of φ, Σ and Δ have the same cardinality. Then, an important question is the following: when $\varphi \in \mathbf{FOR}_n$ is n-unifiable, is there a minimal n-complete set of n-unifiers of φ? When the answer is "yes", how large is this set? For all n-unifiable $\varphi \in \mathbf{FOR}_n$, we say that:

- φ is *n-nullary* if there exists no minimal complete set of unifiers of φ,
- φ is *n-infinitary* if there exists a minimal complete set of unifiers of φ with infinite cardinality,
- φ is *n-finitary* if there exists a minimal complete set of unifiers of φ with finite cardinality ≥ 2,
- φ is *n-unitary* if there exists a minimal complete set of unifiers of φ with cardinality 1.

Obviously, considered as an n-formula, $\Diamond x_1 \to \Box x_1$ is n-unifiable. Indeed, let $(n, \upsilon_\bot)$ and $(n, \upsilon_\top)$ be the n-substitutions defined by:

- $\upsilon_\bot(x_1) = \bot$ and $\upsilon_\top(x_1) = \top$,
- for all $i \in \{2, \ldots, n\}$, $\upsilon_\bot(x_i) = x_i$ and $\upsilon_\top(x_i) = x_i$.

Obviously, $\upsilon_\bot(\Diamond x_1 \to \Box x_1) \in \mathbf{L}_2$ and $\upsilon_\top(\Diamond x_1 \to \Box x_1) \in \mathbf{L}_2$. Hence, $(n, \upsilon_\bot)$ and $(n, \upsilon_\top)$ are n-unifiers of $\Diamond x_1 \to \Box x_1$. Moreover,

Proposition 12 *The n-unifiable n-formula $\Diamond x_1 \to \Box x_1$ is n-finitary.*

For all n-unifiable $\varphi \in \mathbf{FOR}_n$ and for all $\pi \geq 1$, we say that φ is *n-π-reasonable* if for all n-unifiers (k, σ) of φ, if $k \geq \pi$ then there exists an n-unifier (l, τ) of φ such that $(l, \tau) \preccurlyeq_n (k, \sigma)$ and $l \leq \pi$. The idea behind the concept of reasonableness is simple: a unifiable formula is reasonable when a bounded set of variables suffices to express the set of all its unifiers. Since $\mathbf{L}_2$ is locally tabular, it is not particularly surprising that we have the following result.

Proposition 13 *Let $\varphi \in \mathbf{FOR}_n$ be n-unifiable and $\pi \geq 1$. If φ is n-π-reasonable then φ is either n-finitary, or n-unitary.*

As a result, in order to prove that n-unifiable n-formulas are either n-finitary, or n-unitary, it suffices to prove that n-unifiable n-formulas are n-n-reasonable. This is what Proposition 14 asserts below.

6 Main result

Let $n \geq 1$. Our aim is now to prove that n-unifiable n-formulas do not require more variables than the variables $x_1, \ldots, x_n$ in order to express their unifiers. Let $\varphi \in \mathbf{FOR}_n$. Suppose φ is n-unifiable. Let (k, σ) be an n-unifier of φ such that $k \geq n$. To achieve our aim, it suffices to construct an n-unifier (n, τ) of φ such that $(n, \tau) \preccurlyeq_n (k, \sigma)$. The construction of (n,τ) is based on the definition of a surjective (k,n)-morphism f such that for all $(\alpha,S),(\beta,T)\in\mathbf{MOD}_k$, if $f(\alpha,S)=f(\beta,T)$ then $g_{(k,\sigma)}(\alpha,S)=g_{(k,\sigma)}(\beta,T)$. See Lemmas 4, 5 and 6 below. The argument leading to the definition of f is based on a sequence of combinatorial facts. See Lemmas 1, 2 and 3 below. In this argument, Proposition 1 is used twice. Now, let us start. Let $\sim_k$ be the equivalence relation on $\mathbf{MOD}_k$ defined by:

- $(\alpha,S)\sim_k(\beta,T)$ iff $g_{(k,\sigma)}(\alpha,S)=g_{(k,\sigma)}(\beta,T)$.

Lemma 1 1. $\|\mathbf{MOD}_k^{\mathtt{deg}}/\sim_k\|\leq\|\mathbf{MOD}_n^{\mathtt{deg}}\|$,
2. $\|\mathbf{MOD}_n^{\mathtt{deg}}\|\leq\|\mathbf{MOD}_k^{\mathtt{deg}}\|$.

Hence, by Proposition 1 and Lemma 1, there exists a surjective function $f^{\mathtt{deg}}$ from $\mathbf{MOD}_k^{\mathtt{deg}}$ to $\mathbf{MOD}_n^{\mathtt{deg}}$ such that for all $(\alpha,\emptyset),(\beta,\emptyset)\in\mathbf{MOD}_k^{\mathtt{deg}}$, if $f^{\mathtt{deg}}(\alpha,\emptyset)=f^{\mathtt{deg}}(\beta,\emptyset)$ then $(\alpha,\emptyset)\sim_k(\beta,\emptyset)$.

Lemma 2 *For all nonempty sets S,T of k-tuples of bits, if the images by $f^{\mathtt{deg}}$ of $S\times\{\emptyset\}$ and $T\times\{\emptyset\}$ are equal then the images by $g_{(k,\sigma)}$ of $S\times\{\emptyset\}$ and $T\times\{\emptyset\}$ are equal.*

For all nonempty sets E of n-tuples of bits, let

- $f^{\circ}(E)$ be the set of all $(\alpha,S)\in\mathbf{MOD}_k\setminus\mathbf{MOD}_k^{\mathtt{deg}}$ such that the image by $f^{\mathtt{deg}}$ of $S\times\{\emptyset\}$ is equal to $E\times\{\emptyset\}$,
- $f^{\bullet}(E)$ be the set of all $(\alpha,S)\in\mathbf{MOD}_n\setminus\mathbf{MOD}_n^{\mathtt{deg}}$ such that $S=E$.

Notice that for all nonempty sets E of n-tuples of bits, since $f^{\mathtt{deg}}$ is surjective, $\|f^{\circ}(E)\|\geq2^k$. Notice also that for all nonempty sets E of n-tuples of bits, $\|f^{\bullet}(E)\|=2^n$.

Lemma 3 *For all nonempty sets E of n-tuples of bits,*

1. $\|f^{\circ}(E)/\sim_k\|\leq\|f^{\bullet}(E)\|$,
2. $\|f^{\bullet}(E)\|\leq\|f^{\circ}(E)\|$.

Thus, for all nonempty sets E of n-tuples of bits, by Proposition 1 and Lemma 3, there exists a surjective function f^E from $f^{\circ}(E)$ to $f^{\bullet}(E)$ such that for all $(\alpha,S),(\beta,T)\in f^{\circ}(E)$, if $f^E(\alpha,S)=f^E(\beta,T)$ then $(\alpha,S)\sim_k(\beta,T)$. Let f be the function from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ such that for all $(\alpha,\emptyset)\in\mathbf{MOD}_k^{\mathtt{deg}}$,

- $f(\alpha,\emptyset)=f^{\mathtt{deg}}(\alpha,\emptyset)$

and for all $(\alpha,S)\in\mathbf{MOD}_k\setminus\mathbf{MOD}_k^{\mathtt{deg}}$, E being the nonempty set of n-tuples of bits such that the image by $f^{\mathtt{deg}}$ of $S\times\{\emptyset\}$ is equal to $E\times\{\emptyset\}$,

- $f(\alpha,S)=f^E(\alpha,S)$.

Lemma 4 *f is surjective.*

Lemma 5 *f is a (k,n)-morphism.*

Lemma 6 *For all $(\alpha,S),(\beta,T)\in\mathbf{MOD}_k$, if $f(\alpha,S)=f(\beta,T)$ then $g_{(k,\sigma)}(\alpha,S)=g_{(k,\sigma)}(\beta,T)$.*

The reader is invited to notice how Proposition 1 has been used — twice — in the argument leading to the definition of f. Let (n, τ), (k, ν) be the n-substitutions defined by:

- $\tau(x_i) = \bigvee\{\mathbf{for}_n(f(\alpha, S)) : (\alpha, S) \in \mathbf{MOD}_k$ is such that $(\alpha, S) \models_k \sigma(x_i)\}$ where $i \in \{1, \ldots, n\}$,
- $\nu(x_i) = \bigvee\{\mathbf{for}_k(\alpha, S) : (\alpha, S) \in \mathbf{MOD}_k$ is such that $f(\alpha, S) \models_n x_i\}$ where $i \in \{1, \ldots, n\}$.

In Lemmas 7–10 below, we prove interesting properties of these n-substitutions: Lemma 7 is used for showing that (n, τ) is an n-unifier of φ, Lemma 8 is used in the proof of Lemma 9, Lemma 9 is used in the proof of Lemma 10 and Lemma 10 is used for showing that $(n, \tau) \preccurlyeq_n (k, \sigma)$.

Lemma 7 *Let $\psi \in \mathbf{FOR}_n$. For all $(\beta, T) \in \mathbf{MOD}_n$, the following conditions are equivalent: (i) there exists $(\alpha, S) \in \mathbf{MOD}_k$ such that $f(\alpha, S) = (\beta, T)$ and $(\alpha, S) \models_k \sigma(\psi)$; (ii) for all $(\alpha, S) \in \mathbf{MOD}_k$, if $f(\alpha, S) = (\beta, T)$ then $(\alpha, S) \models_k \sigma(\psi)$; (iii) $(\beta, T) \models_n \tau(\psi)$.*

Lemma 8 *For all $(\beta, T) \in \mathbf{MOD}_k$ and for all $i \in \{1, \ldots, n\}$, $(\beta, T) \models_k \nu(x_i)$ iff $f(\beta, T) \models_n x_i$.*

Lemma 9 *For all $(\beta, T) \in \mathbf{MOD}_k$ and for all $(\gamma, U) \in \mathbf{MOD}_n$, $f(\beta, T) = (\gamma, U)$ iff $(\beta, T) \models_k \nu(\mathbf{for}_n(\gamma, U))$.*

Lemma 10 *For all $(\beta, T) \in \mathbf{MOD}_k$ and for all $i \in \{1, \ldots, n\}$, $(\beta, T) \models_k \nu(\tau(x_i))$ iff $(\beta, T) \models_k \sigma(x_i)$.*

Since (k, σ) is an n-unifier of φ, $\sigma(\varphi) \in \mathbf{L}_2$. Thus, by Proposition 3, $\sigma(\varphi)$ is k-valid, i.e. for all $(\alpha, S) \in \mathbf{MOD}_k$, $(\alpha, S) \models_k \sigma(\varphi)$. Consequently, by Lemma 7, for all $(\beta, T) \in \mathbf{MOD}_n$, $(\beta, T) \models_n \tau(\varphi)$, i.e. $\tau(\varphi)$ is n-valid. Hence, by Proposition 3, $\tau(\varphi) \in \mathbf{L}_2$. Thus, (n, τ) is an n-unifier of φ. Since by Lemma 10, $(n, \tau) \preccurlyeq_n (k, \sigma)$, φ is n-n-reasonable. Since $\varphi \in \mathbf{FOR}_n$ is arbitrary, this proves the following result.

Proposition 14 *For all $\varphi \in \mathbf{FOR}_n$, if φ is n-unifiable then φ is n-n-reasonable.*

Propositions 13 and 14, lead us to the following result.

Theorem 1 *For all $\varphi \in \mathbf{FOR}_n$, if φ is n-unifiable then φ is either n-finitary, or n-unitary.*

In the light of Proposition 12 and Theorem 1, we therefore conclude that $\mathbf{L}_2$ is finitary.

7 Conclusion

In this paper, within the context of elementary unification, we have proved Theorem 1 asserting that in $\mathbf{L}_2$, unifiable formulas are either finitary, or unitary. We believe that in the line of reasoning leading to Theorem 1, the main properties of $\mathbf{L}_2$ are the ones given in Propositions 2, 6 and 7. Proposition 2 says that $\mathbf{L}_2$ is locally tabular — it is used in the proof of Proposition 13. For all $n \geq 1$, Propositions 6 and 7 tell us how to associate a function $g_{(k,\sigma)}$ from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ to any n-substitution (k, σ) — they are used in the proof of Proposition 14. Notice that Theorem 1 is an immediate consequence of Propositions 13 and 14. However, since $\mathbf{L}_2$ is locally tabular, the reader may think that the line of reasoning leading to Theorem 1 is unnecessarily complicated. In other respect, since the unification type is a categorical invariant, the reader may think that it is possible to directly work in the category of all finite models of $\mathbf{L}_2$. Indeed, the categorical approach to the unification problem in propositional logic is powerful [18]. Nevertheless, the fact that the question of the unification type of $\mathbf{L}_2$ has not been solved before indicates that things may not be so simple.

Here are open questions: (i) determine the unification type of $\mathbf{K} + \Box^d\bot$ (the least modal logic containing $\Box^d\bot$) for each $d\geq 3$, (ii) determine the unification types of the locally tabular modal logics studied in [26–28], (iii) determine the unification types of **KB**, **KD** and **KT**. We conjecture that the modal logics mentioned in (i) and (ii) are either finitary, or unitary. As for the modal logics considered in (iii), it is only known that **KD** and **KT** are not unitary within the context of elementary unification and **KB**, **KD** and **KT** are nullary within the context of unification with constants [4, 5, 7]. We conjecture that the modal logics mentioned in (iii) are finitary within the context of elementary unification. And of course there is also the open question of the decidability of the unification problem in the modal logics considered in (i), (ii) and (iii). When the modal logics mentioned in (i) and (ii) are decidable, their local tabularity implies the decidability of their unification problem within the context of elementary unification. Concerning the modal logics considered in (iii), the decidability of their unification problem remains a mystery within the context of unification with constants.[6]

Recently, the question of the unification type has been considered within the context of a semantic restriction of description logic $\mathcal{FL}_0$. The formulas of $\mathcal{FL}_0$ are constructed by means of the connectives $\top$, $\wedge$ and $\Box_a$ — where a ranges over a countably infinite set **ACT**. The unification problem in $\mathcal{FL}_0$ is to determine, given a couple (φ, ψ) of formulas, whether there exists a substitution σ such that $\sigma(\varphi)$ and $\sigma(\psi)$ are logically equivalent in the class of all **ACT**-frames, i.e. Kripke frames of the form (W, R) where W is a nonempty set and R_a is a binary relation on W for each $a\in$**ACT**. As is well-known, the unification type of $\mathcal{FL}_0$ is nullary [3]. Restricting the discussion to the class of all **ACT**-frames (W, R) such that for all $a, b\in$**ACT**, $R_a \circ R_b=\emptyset$, Baader et al. [1] have proved that the unification type of $\mathcal{FL}_0$ is unitary for elementary unification and finitary for unification with constants. Now, interpreting the formulas constructed by means of the connectives $\bot$, $\neg$, $\vee$ and $\Box_a$ — where a ranges over **ACT** — in the class of all **ACT**-frames (W, R) such that for all $a, b\in$**ACT**, $R_a \circ R_b=\emptyset$, we conjecture that the unifiable ones are either finitary, or unitary.

Appendix

Proof of Proposition 1: Suppose S is nonempty. Let $\sim$ be an equivalence relation on S.

Suppose $\|S/\!\sim\!\|\leq\|T\|\leq\|S\|$. Let h be a function from $S/\!\sim$ to S such that for all $\alpha\in S$, $h([\alpha])\in[\alpha]$, i.e. h is a function selecting an element in each equivalence class modulo $\sim$.[7] Obviously, h is injective. Let $S_0=\{h([\alpha]) : \alpha\in S\}$. Since h is injective, $\|S/\!\sim\!\|=\|S_0\|$. Since $\|S/\!\sim\!\|\leq\|T\|$, $\|S_0\|\leq\|T\|$. Let T_0 be a subset of T such that $\|T_0\|=\|S_0\|$. Let f_0 be a one-to-one correspondence between S_0 and T_0. Let $T_1=T\backslash T_0$. Notice that T_0 and T_1 make a partition of T. Since $\|T\|\leq\|S\|$ and $\|T_0\|=\|S_0\|$, $\|T_1\|\leq\|S\backslash S_0\|$. Let S_1 be a subset of $S\backslash S_0$ such that $\|S_1\|=\|T_1\|$. Let f_1 be a one-to-one correspondence between S_1 and T_1. Let $S_2=(S\backslash S_0)\backslash S_1$. Let f_2 be the function from S_2 to T such that for all $\alpha\in S_2$,

[6] As is well-known, the unification problem in **KB**, **KD** and **KT** is in **NP** within the context of elementary unification.

[7] Since S is finite, the proof of the existence of $\sim$ does not require the use of the axiom of choice.

$f_2(\alpha)=f_0(h([\alpha]))$. Let f be the function from S to T defined by $f|S_0=f_0$, $f|S_1=f_1$ and $f|S_2=f_2$. □

Claim *f is surjective.*

Proof Let $\beta\in T$. We consider the following cases. □

Case $\beta\in T_0$. Since f_0 is one-to-one, let $\alpha\in S_0$ be such that $f_0(\alpha)=\beta$. Thus, $\alpha\in S$. Moreover, $f(\alpha)=f_0(\alpha)$. Since $f_0(\alpha)=\beta$, $f(\alpha)=\beta$.

Case $\beta\in T_1$. Since f_1 is one-to-one, let $\alpha\in S_1$ be such that $f_1(\alpha)=\beta$. Hence, $\alpha\in S$. Moreover, $f(\alpha)=f_1(\alpha)$. Since $f_1(\alpha)=\beta$, $f(\alpha)=\beta$.

Claim *For all $\alpha,\beta\in S$, if $f(\alpha)=f(\beta)$ then $\alpha\sim\beta$.*

Proof Let $\alpha,\beta\in S$ be such that $f(\alpha)=f(\beta)$. We consider the following cases.

Case $\alpha\in S_0$ and $\beta\in S_0$. Consequently, $f(\alpha)=f_0(\alpha)$ and $f(\beta)=f_0(\beta)$. Since $f(\alpha)=f(\beta)$, $f_0(\alpha)=f_0(\beta)$. Since f_0 is one-to-one, $\alpha=\beta$. Thus, $\alpha\sim\beta$.

Case $\alpha\in S_0$ and $\beta\in S_1$. Consequently, $f(\alpha)=f_0(\alpha)$ and $f(\beta)=f_1(\beta)$. Since $f(\alpha)=f(\beta)$, $f_0(\alpha)=f_1(\beta)$. Since $f_0(\alpha)\in T_0$ and $f_1(\beta)\in T_1$, T_0 and T_1 do not make a partition of T: a contradiction.

Case $\alpha\in S_0$ and $\beta\in S_2$. Hence, $f(\alpha)=f_0(\alpha)$ and $f(\beta)=f_2(\beta)$. Since $f(\alpha)=f(\beta)$, $f_0(\alpha)=f_2(\beta)$. Thus, $f_0(\alpha)=f_0(h([\beta]))$. Since f_0 is one-to-one, $\alpha=h([\beta])$. Since $h([\beta])\in[\beta]$, $\alpha\in[\beta]$. Consequently, $\alpha\sim\beta$.

Case $\alpha\in S_1$ and $\beta\in S_1$. Hence, $f(\alpha)=f_1(\alpha)$ and $f(\beta)=f_1(\beta)$. Since $f(\alpha)=f(\beta)$, $f_1(\alpha)=f_1(\beta)$. Since f_1 is one-to-one, $\alpha=\beta$. Thus, $\alpha\sim\beta$.

Case $\alpha\in S_1$ and $\beta\in S_2$. Hence, $f(\alpha)=f_1(\alpha)$ and $f(\beta)=f_2(\beta)$. Since $f(\alpha)=f(\beta)$, $f_1(\alpha)=f_2(\beta)$. Thus, $f_1(\alpha)=f_0(h([\beta]))$. Since $f_1(\alpha)\in T_1$ and $f_0(h([\beta]))\in T_0$, T_0 and T_1 do not make a partition of T: a contradiction.

Case $\alpha\in S_2$ and $\beta\in S_2$. Hence, $f(\alpha)=f_2(\alpha)$ and $f(\beta)=f_2(\beta)$. Since $f(\alpha)=f(\beta)$, $f_2(\alpha)=f_2(\beta)$. Consequently, $f_0(h([\alpha]))=f_0(h([\beta]))$. Since f_0 is one-to-one, $h([\alpha])=h([\beta])$. Since $h([\alpha])\in[\alpha]$ and $h([\beta])\in[\beta]$, $[\alpha]\cap[\beta]\neq\emptyset$. Thus, $\alpha\sim\beta$.

Suppose f is a surjective function from S to T such that for all $\alpha,\beta\in S$, if $f(\alpha)=f(\beta)$ then $\alpha\sim\beta$. For the sake of the contradiction, suppose either $\|S/\sim\|>\|T\|$, or $\|T\|>\|S\|$. Since f is surjective, $\|T\|\leq\|S\|$. Since either $\|S/\sim\|>\|T\|$, or $\|T\|>\|S\|$, $\|S/\sim\|>\|T\|$. Let $p\in\mathbb{N}$ and $\beta^1,\ldots,\beta^p\in S$ be such that $p>\|T\|$ and for all $q,r\in\mathbb{N}$, if $1\leq q,r\leq p$ and $q\neq r$ then $\beta^q\not\sim\beta^r$. Hence, for all $q,r\in\mathbb{N}$, if $1\leq q,r\leq p$ and $q\neq r$ then $f(\beta^q)\neq f(\beta^r)$. Thus, $p\leq\|T\|$: a contradiction.

□

Proof of Proposition 2: From [11, Proposition 2.29] and from the fact that for all $\varphi\in\mathbf{FOR}_n$, there exists $\psi\in\mathbf{FOR}_n$ such that $\mathtt{deg}_n(\psi)<2$ and $\varphi\leftrightarrow\psi\in\mathbf{L}_2$. □

Proof of Proposition 3: From [11, Proposition 2.6], from [11, Lemma 4.21] and from the fact that for all maximal consistent sets w of n-formulas, there exists $(\alpha,S)\in\mathbf{MOD}_n$ such that the submodel of the canonical model of $\mathbf{L}_2$ generated by w is isomorphic to (α,S). □

Proof of Proposition 4: Let $(\alpha, S)\in\mathbf{MOD}_n$ and $\beta\in\mathbf{BIT}_n$. For the sake of the contradiction, suppose $\alpha=\beta$ not-iff $(\alpha, S)\models_n\bar{x}^{\beta}$. Hence, either $\alpha=\beta$ and $(\alpha, S)\not\models_n\bar{x}^{\beta}$, or $\alpha\neq\beta$ and $(\alpha, S)\models_n\bar{x}^{\beta}$. In the former case, for all $i\in\{1,\dots,n\}$, $\alpha_i=\beta_i$. Thus, for all $i\in\{1,\dots,n\}$, $(\alpha, S)\models_n x_i^{\beta_i}$. Consequently, $(\alpha, S)\models_n\bar{x}^{\beta}$: a contradiction. In the latter case, let $i\in\{1,\dots,n\}$ be such that $\alpha_i\neq\beta_i$. Hence, $(\alpha, S)\not\models_n x_i^{\beta_i}$. Thus, $(\alpha, S)\not\models_n\bar{x}^{\beta}$: a contradiction. □

Proof of Proposition 5: From [20, Theorem 32], from the fact that for all $(\alpha, S)\in\mathbf{MOD}_n$, $\mathbf{for}_n(\alpha, S)$ characterizes n-models modulo bisimulation and from the fact that for all $(\alpha, S), (\beta, T)\in\mathbf{MOD}_n$, if (α, S) and (β, T) are bisimilar then $(\alpha, S)=(\beta, T)$. □

Proof of Proposition 6: Let $(\alpha, S)\in\mathbf{MOD}_k$. Let β be the n-tuple of bits such that for all $i\in\{1,\dots,n\}$, if $(\alpha, S)\not\models_k\sigma(x_i)$ then $\beta_i=0$ else $\beta_i=1$. Let T be the least set of n-tuples of bits such that for all $\gamma\in S$, there exists $\delta\in T$ such that for all $i\in\{1,\dots,n\}$, if $(\gamma,\emptyset)\not\models_k\sigma(x_i)$ then $\delta_i=0$ else $\delta_i=1$. By induction on $\varphi\in\mathbf{FOR}_n$, the reader may easily verify that $(\alpha, S)\models_k\sigma(\varphi)$ iff $(\beta, T)\models_n\varphi$. Since by Proposition 5, $(\beta, T)\models_n\mathbf{for}_n(\beta, T)$, $(\alpha, S)\models_k\sigma(\mathbf{for}_n(\beta, T))$. □

Proof of Proposition 7: Let $(\beta, T), (\gamma, U)\in\mathbf{MOD}_n$. Suppose $(\alpha, S)\models_k\sigma(\mathbf{for}_n(\beta, T))$ and $(\alpha, S)\models_k\sigma(\mathbf{for}_n(\gamma, U))$. Hence, $(\alpha, S)\models_k\sigma(\bar{x}^{\beta})$ and $(\alpha, S)\models_k\sigma(\bar{x}^{\gamma})$. Thus, $\beta=\gamma$. Let $\beta'\in T$ be arbitrary. Since $(\alpha, S)\models_k\sigma(\mathbf{for}_n(\beta, T))$, let $\alpha'\in S$ be such that $(\alpha',\emptyset)\models_k\sigma(\bar{x}^{\beta'})$. Since $(\alpha, S)\models_k\sigma(\mathbf{for}_n(\gamma, U))$, let $\gamma'\in U$ be such that $(\alpha',\emptyset)\models_k\sigma(\bar{x}^{\gamma'})$. Since $(\alpha',\emptyset)\models_k\sigma(\bar{x}^{\beta'})$, $\beta'=\gamma'$. Consequently, $\beta'\in U$. Since β' is arbitrary, $T\subseteq U$. Reciprocally, the reader may easily verify that $U\subseteq T$. Hence, $T=U$. Since $\beta=\gamma$, $(\beta, T)=(\gamma, U)$. □

Proof of Proposition 8: Let $(k,\sigma)\in\mathbf{SUB}_n$. For the sake of the contradiction, suppose $g_{(k,\sigma)}$ is not a (k,n)-morphism. Hence, let $(\alpha, S)\in\mathbf{MOD}_k$ and $(\beta, T)\in\mathbf{MOD}_n$ be such that $g_{(k,\sigma)}(\alpha, S)=(\beta, T)$ — and therefore $(\alpha, S)\models_k\sigma(\mathbf{for}_n(\beta, T))$ — and either there exists $\gamma\in S$ such that for all $\delta\in T$, $g_{(k,\sigma)}(\gamma,\emptyset)\neq(\delta,\emptyset)$, or there exists $\delta\in T$ such that for all $\gamma\in S$, $g_{(k,\sigma)}(\gamma,\emptyset)\neq(\delta,\emptyset)$. In the former case, let $\delta'\in T$ be such that $(\gamma,\emptyset)\models_k\sigma(\bar{x}^{\delta'})$. Thus, $g_{(k,\sigma)}(\gamma,\emptyset)\neq(\delta',\emptyset)$. Since $(\gamma,\emptyset)\models_k\sigma(\bar{x}^{\delta'})$, $g_{(k,\sigma)}(\gamma,\emptyset)=(\delta',\emptyset)$: a contradiction. In the latter case, let $\gamma'\in S$ be such that $(\gamma',\emptyset)\models_k\sigma(\bar{x}^{\delta})$. Consequently, $g_{(k,\sigma)}(\gamma',\emptyset)\neq(\delta,\emptyset)$. Since $(\gamma',\emptyset)\models_k\sigma(\bar{x}^{\delta})$, $g_{(k,\sigma)}(\gamma',\emptyset)=(\delta,\emptyset)$: a contradiction. □

Proof of Proposition 9: Let $(k,\sigma)\in\mathbf{SUB}_n$ and $(\alpha, S), (\beta, T)\in\mathbf{MOD}_k$. Suppose $g_{(k,\sigma)}(\alpha, S)=g_{(k,\sigma)}(\beta, T)$. Hence, let $(\gamma, U)\in\mathbf{MOD}_n$ be such that $g_{(k,\sigma)}(\alpha, S)=(\gamma, U)$ and $g_{(k,\sigma)}(\beta, T)=(\gamma, U)$. Thus, $(\alpha, S)\models_k\sigma(\mathbf{for}_n(\gamma, U))$ and $(\beta, T)\models_k\sigma(\mathbf{for}_n(\gamma, U))$. Consequently, $(\alpha, S)\models_k\sigma(\bar{x}^{\gamma})$ and $(\beta, T)\models_k\sigma(\bar{x}^{\gamma})$. Hence, for all $i\in\{1,\dots,n\}$, $(\alpha, S)\models_k\sigma(x_i)$ iff $(\beta, T)\models_k\sigma(x_i)$. □

Proof of Proposition 10: For the sake of the contradiction, suppose $f(\beta, T)=(\gamma, U)$ and the image by f of $T\times\{\emptyset\}$ is not equal to $U\times\{\emptyset\}$. Hence, either the image by f of $T\times\{\emptyset\}$ is not included in $U\times\{\emptyset\}$, or the image by f of $T\times\{\emptyset\}$ does not include to $U\times\{\emptyset\}$. In the former case, let $\delta\in T$ be such that $f(\delta,\emptyset)\notin U\times\{\emptyset\}$. Since f is a (k,n)-morphism and $f(\beta, T)=(\gamma, U)$, let $\epsilon'\in U$ be such that $f(\delta,\emptyset)=(\epsilon',\emptyset)$. Thus, $f(\delta,\emptyset)\in U\times\{\emptyset\}$: a contradiction. In the latter case, let $\epsilon\in U$ be such that $(\epsilon,\emptyset)\notin f(T\times\{\emptyset\})$. Since f is a

(k,n)-morphism and $f(\beta,T)=(\gamma,U)$, let $\delta'\in T$ be such that $f(\delta',\emptyset)=(\epsilon,\emptyset)$. Consequently, $(\epsilon,\emptyset)\in f(T\times\{\emptyset\})$: a contradiction. □

Proof of Proposition 11: For the sake of the contradiction, suppose $f(\beta,T)\neq(\gamma,U)$, $f(\beta,T)\models_n\bar{x}^\gamma$, for all $\delta\in T$, there exists $\epsilon\in U$ such that $f(\delta,\emptyset)=(\epsilon,\emptyset)$ and for all $\epsilon\in U$, there exists $\delta\in T$ such that $f(\delta,\emptyset)=(\epsilon,\emptyset)$. Let $(\gamma',U')\in\mathbf{MOD}_n$ be such that $f(\beta,T)=(\gamma',U')$. Since $f(\beta,T)\models_n\bar{x}^\gamma$, by Proposition 4, $\gamma=\gamma'$. Since $f(\beta,T)\neq(\gamma,U)$ and $f(\beta,T)=(\gamma',U')$, $U\neq U'$. Hence, either $U\not\subseteq U'$, or $U\not\supseteq U'$. In the former case, let $\epsilon'\in U$ be such that $\epsilon'\notin U'$. Since for all $\epsilon\in U$, there exists $\delta\in T$ such that $f(\delta,\emptyset)=(\epsilon,\emptyset)$, let $\delta'\in T$ be such that $f(\delta',\emptyset)=(\epsilon',\emptyset)$. Since f is a (k,n)-morphism and $f(\beta,T)=(\gamma',U')$, $\epsilon'\in U'$: a contradiction. In the latter case, let $\epsilon''\in U'$ be such that $\epsilon''\notin U$. Since f is a (k,n)-morphism and $f(\beta,T)=(\gamma',U')$, let $\delta''\in T$ be such that $f(\delta'',\emptyset)=(\epsilon'',\emptyset)$. Since for all $\delta\in T$, there exists $\epsilon\in U$ such that $f(\delta,\emptyset)=(\epsilon,\emptyset)$, $\epsilon''\in U$: a contradiction. □

Proof of Proposition 12: Let (n,σ) and (n,τ) be the n-substitutions defined by:

- $\sigma(x_1)=\Box\bot\vee x_1$ and $\tau(x_1)=\Diamond\top\wedge x_1$,
- for all $i\in\{2,\ldots,n\}$, $\sigma(x_i)=x_i$ and $\tau(x_i)=x_i$.

Obviously, $\sigma(\Diamond x_1\rightarrow\Box x_1)\in\mathbf{L}_2$ and $\tau(\Diamond x_1\rightarrow\Box x_1)\in\mathbf{L}_2$. Hence, (n,σ) and (n,τ) are n-unifiers of $\Diamond x_1\rightarrow\Box x_1$. In order to prove that $\Diamond x_1\rightarrow\Box x_1$ is n-finitary, it suffices to prove that $\{(n,\sigma),(n,\tau)\}$ is a minimal n-complete set of n-unifiers of $\Diamond x_1\rightarrow\Box x_1$. □

n-completeness of $\{(n,\sigma),(n,\tau)\}$: Let (k,υ) be an arbitrary n-unifier of $\Diamond x_1\rightarrow\Box x_1$. Thus, $\upsilon(\Diamond x_1\rightarrow\Box x_1)\in\mathbf{L}_2$. By using the semantics of $\mathbf{L}_2$, it follows that either $\Box\bot\rightarrow\upsilon(x_1)\in\mathbf{L}_2$, or $\upsilon(x_1)\rightarrow\Diamond\top\in\mathbf{L}_2$. Indeed, for the sake of the contradiction, suppose neither $\Box\bot\rightarrow\upsilon(x_1)\in\mathbf{L}_2$, nor $\upsilon(x_1)\rightarrow\Diamond\top\in\mathbf{L}_2$. Consequently, by Proposition 3, neither $\Box\bot\rightarrow\upsilon(x_1)$ is k-valid, nor $\upsilon(x_1)\rightarrow\Diamond\top$ is k-valid. Hence, let $(\alpha,\emptyset),(\beta,\emptyset)\in\mathbf{MOD}_k^{\mathsf{deg}}$ be such that $(\alpha,\emptyset)\not\models_k\upsilon(x_1)$ and $(\beta,\emptyset)\models_k\upsilon(x_1)$. Let $\gamma\in\mathbf{BIT}_k$. Since $(\alpha,\emptyset)\not\models_k\upsilon(x_1)$ and $(\beta,\emptyset)\models_k\upsilon(x_1)$, $(\gamma,\{\alpha,\beta\})\not\models_k\upsilon(\Diamond x_1\rightarrow\Box x_1)$. Thus, $\upsilon(\Diamond x_1\rightarrow\Box x_1)$ is not k-valid. Consequently, by Proposition 3, $\upsilon(\Diamond x_1\rightarrow\Box x_1)\notin\mathbf{L}_2$: a contradiction. In the former case where $\Box\bot\rightarrow\upsilon(x_1)\in\mathbf{L}_2$, it follows immediately that $\upsilon(\sigma(x_1))\equiv_k\upsilon(x_1)$. Hence, $(n,\sigma)\preccurlyeq_n(k,\upsilon)$. In the latter case where $\upsilon(x_1)\rightarrow\Diamond\top\in\mathbf{L}_2$, it follows immediately that $\upsilon(\tau(x_1))\equiv_{\mathbf{L}_2}\upsilon(x_1)$. Thus, $(n,\tau)\preccurlyeq_n(k,\upsilon)$.

Minimality of $\{(n,\sigma),(n,\tau)\}$: For the sake of the contradiction, suppose $\{(n,\sigma),(n,\tau)\}$ is not minimal. Consequently, either $(n,\sigma)\preccurlyeq_n(n,\tau)$, or $(n,\tau)\preccurlyeq_n(n,\sigma)$. In the former case, there exists an n-substitution (n,υ) such that $\upsilon(\sigma(x_1))\equiv_n\tau(x_1)$. Hence, $\Box\bot\vee\upsilon(x_1)\equiv_n\Diamond\top\wedge x_1$. In the latter case, there exists a substitution (n,υ) such that $\upsilon(\tau(x_1))\equiv_n\sigma(x)$. Thus, $\Diamond\top\wedge\upsilon(x_1)\equiv_n\Box\bot\vee x_1$. In both cases, $\Box\bot\rightarrow\Diamond\top\in\mathbf{L}_2$. Consequently, $\Diamond\top\in\mathbf{L}_2$: a contradiction.

Proof of Proposition 13: Suppose φ is n-π-reasonable. Let Σ be the set of all n-unifiers of φ. Notice that Σ is n-complete. Let Σ' be the set of n-substitutions obtained from Σ by keeping only the n-substitutions (k,σ) such that $k\leq\pi$. Since φ is n-π-reasonable and Σ is n-complete, Σ' is n-complete. Let Σ'' be the set of n-substitutions obtained from Σ' by keeping only one representative of each equivalence class modulo $\simeq_n$. Since Σ' is n-complete, Σ'' is n-complete. Moreover, by Proposition 2, Σ'' is finite. Hence, either φ is n-finitary, or φ is n-unitary. □

Proof of Lemma 1: (**1**) For the sake of the contradiction, suppose $\|\mathbf{MOD}_k^{\mathsf{deg}}/{\sim_k}\|>\|\mathbf{MOD}_n^{\mathsf{deg}}\|$. Let $p\in\mathbb{N}$ and $(\alpha^1,\emptyset),\ldots,(\alpha^p,\emptyset)\in\mathbf{MOD}_k^{\mathsf{deg}}$ be such that $p>\|\mathbf{MOD}_n^{\mathsf{deg}}\|$ and for all $q,r\in\mathbb{N}$, if $1\leq q,r\leq p$ and $q\neq r$ then $(\alpha^q,\emptyset)\not\sim_k(\alpha^r,\emptyset)$. Hence, for all $q,r\in\mathbb{N}$, if $1\leq q,r\leq p$ and $q\neq r$ then $g_{(k,\sigma)}(\alpha^q,\emptyset)\neq g_{(k,\sigma)}(\alpha^r,\emptyset)$. Since $g_{(k,\sigma)}$ is a (k,n)-morphism, let $\beta^1,\ldots,\beta^p\in\mathbf{BIT}_n$ be such that $g_{(k,\sigma)}(\alpha^1,\emptyset)=(\beta^1,\emptyset)$, ..., $g_{(k,\sigma)}(\alpha^p,\emptyset)=(\beta^p,\emptyset)$. Since for all $q,r\in\mathbb{N}$, if $1\leq q,r\leq p$ and $q\neq r$ then $g_{(k,\sigma)}(\alpha^q,\emptyset)\neq g_{(k,\sigma)}(\alpha^r,\emptyset)$, for all $q,r\in\mathbb{N}$, if $1\leq q,r\leq p$ and $q\neq r$ then $\beta^q\neq\beta^r$. Thus, $p\leq 2^n$. Since $\|\mathbf{MOD}_n^{\mathsf{deg}}\|=2^n$, $p\leq\|\mathbf{MOD}_n^{\mathsf{deg}}\|$: a contradiction. □

(**2**) Since $\|\mathbf{MOD}_n^{\mathsf{deg}}\|=2^n$, $\|\mathbf{MOD}_k^{\mathsf{deg}}\|=2^k$ and $k\geq n$, $\|\mathbf{MOD}_n^{\mathsf{deg}}\|\leq\|\mathbf{MOD}_k^{\mathsf{deg}}\|$.

Proof of Lemma 2: Let S,T be nonempty sets of k-tuples of bits. Suppose the images by f^{deg} of $S\times\{\emptyset\}$ and $T\times\{\emptyset\}$ are equal. For the sake of the contradiction, suppose the images by $g_{(k,\sigma)}$ of $S\times\{\emptyset\}$ and $T\times\{\emptyset\}$ are not equal. Since $g_{(k,\sigma)}$ is a (k,n)-morphism, let $(\gamma,\emptyset)\in\mathbf{MOD}_n^{\mathsf{deg}}$ be such that either $(\gamma,\emptyset)$ is in the image by $g_{(k,\sigma)}$ of $S\times\{\emptyset\}$ without being in the image by $g_{(k,\sigma)}$ of $T\times\{\emptyset\}$, or $(\gamma,\emptyset)$ is in the image by $g_{(k,\sigma)}$ of $T\times\{\emptyset\}$ without being in the image by $g_{(k,\sigma)}$ of $S\times\{\emptyset\}$. Without loss of generality, suppose $(\gamma,\emptyset)$ is in the image by $g_{(k,\sigma)}$ of $S\times\{\emptyset\}$ without being in the image by $g_{(k,\sigma)}$ of $T\times\{\emptyset\}$. Hence, let $\alpha\in S$ be such that $g_{(k,\sigma)}(\alpha,\emptyset)=(\gamma,\emptyset)$. Since the images by f^{deg} of $S\times\{\emptyset\}$ and $T\times\{\emptyset\}$ are equal, let $\beta\in T$ be such that $f^{\mathsf{deg}}(\alpha,\emptyset)=f^{\mathsf{deg}}(\beta,\emptyset)$. Thus, $(\alpha,\emptyset)\sim_k(\beta,\emptyset)$. Consequently, $g_{(k,\sigma)}(\alpha,\emptyset)=g_{(k,\sigma)}(\beta,\emptyset)$. Since $g_{(k,\sigma)}(\alpha,\emptyset)=(\gamma,\emptyset)$, $(\gamma,\emptyset)$ is in the image by $g_{(k,\sigma)}$ of $T\times\{\emptyset\}$: a contradiction. □

Proof of Lemma 3: Let E be a nonempty set of n-tuples of bits. (**1**) For the sake of the contradiction, suppose $\|f^{\circ}(E)/{\sim_k}\|>\|f^{\bullet}(E)\|$. Let $p\in\mathbb{N}$ and $(\alpha^1,S_1),\ldots,(\alpha^p,S_p)\in\mathbf{MOD}_k\setminus\mathbf{MOD}_k^{\mathsf{deg}}$ be such that $p>\|f^{\bullet}(E)\|$, the images by f^{deg} of $S_1\times\{\emptyset\},\ldots,S_p\times\{\emptyset\}$ are equal to $E\times\{\emptyset\}$ and for all $q,r\in\mathbb{N}$, if $1\leq q,r\leq p$ and $q\neq r$ then $(\alpha^q,S_q)\not\sim_k(\alpha^r,S_r)$. Hence, for all $q,r\in\mathbb{N}$, if $1\leq q,r\leq p$ and $q\neq r$ then $g_{(k,\sigma)}(\alpha^q,S_q)\neq g_{(k,\sigma)}(\alpha^r,S_r)$. Since $g_{(k,\sigma)}$ is a (k,n)-morphism and the images by f^{deg} of $S_1\times\{\emptyset\},\ldots,S_p\times\{\emptyset\}$ are equal to $E\times\{\emptyset\}$, let $\beta^1,\ldots,\beta^p\in\mathbf{BIT}_n$ be such that $g_{(k,\sigma)}(\alpha^1,S_1)=(\beta^1,E)$, ..., $g_{(k,\sigma)}(\alpha^p,S_p)=(\beta^p,E)$. Since for all $q,r\in\mathbb{N}$, if $1\leq q,r\leq p$ and $q\neq r$ then $g_{(k,\sigma)}(\alpha^q,S_q)\neq g_{(k,\sigma)}(\alpha^r,S_r)$, for all $q,r\in\mathbb{N}$, if $1\leq q,r\leq p$ and $q\neq r$ then $\beta^q\neq\beta^r$. Thus, $p\leq 2^n$. Since $\|f^{\bullet}(E)\|=2^n$, $p\leq\|f^{\bullet}(E)\|$: a contradiction. □

(**2**) Since $k\geq n$, $\|f^{\circ}(E)\|\geq 2^k$ and $\|f^{\bullet}(E)\|=2^n$, $\|f^{\bullet}(E)\|\leq\|f^{\circ}(E)\|$.

Proof of Lemma 4: Let $\beta\in\mathbf{BIT}_n$ and T be a set of n-tuples of bits. We consider the following cases. □

Case $(\beta,T)\in\mathbf{MOD}_n^{\mathsf{deg}}$. Since f^{deg} is surjective, let $\alpha\in\mathbf{BIT}_k$ be such that $f^{\mathsf{deg}}(\alpha,\emptyset)=(\beta,\emptyset)$. Hence, $f(\alpha,\emptyset)=(\beta,T)$.

Case $(\beta,T)\in\mathbf{MOD}_n\setminus\mathbf{MOD}_n^{\mathsf{deg}}$. Thus, $(\beta,T)\in f^{\bullet}(T)$. Since f^T is surjective, let $(\alpha,S)\in f^{\circ}(T)$ be such that $f^T(\alpha,S)=(\beta,T)$. Consequently, $(\alpha,S)\in\mathbf{MOD}_k\setminus\mathbf{MOD}_k^{\mathsf{deg}}$ and the image by f^{deg} of $S\times\{\emptyset\}$ is equal to $T\times\{\emptyset\}$. Hence, $f(\alpha,S)=f^T(\alpha,S)$. Since $f^T(\alpha,S)=(\beta,T)$, $f(\alpha,S)=(\beta,T)$.

Proof of Lemma 5: For the sake of the contradiction, suppose f is not a (k,n)-morphism. Hence, let $(\alpha,S)\in\mathbf{MOD}_k$ and $(\beta,T)\in\mathbf{MOD}_n$ be such that $f(\alpha,S)=(\beta,T)$ and either there exists $\gamma'\in S$ such that for all $\delta'\in T$, $f(\gamma',\emptyset)\neq(\delta',\emptyset)$, or there exists $\delta''\in T$ such that for all $\gamma''\in S$, $f(\gamma'',\emptyset)\neq(\delta'',\emptyset)$. In the former case, $S\neq\emptyset$. Thus, $(\alpha,S)\in\mathbf{MOD}_k\setminus\mathbf{MOD}_k^{\mathsf{deg}}$ and, E

being the nonempty set of n-tuples of bits such that the image by f^{deg} of $S \times \{\emptyset\}$ is equal to $E \times \{\emptyset\}$, $f(\alpha, S)=f^E(\alpha, S)$. Since $f(\alpha, S)=(\beta, T)$, $f^E(\alpha, S)=(\beta, T)$. Consequently, $T=E$. Since the image by f^{deg} of $S \times \{\emptyset\}$ is equal to $E \times \{\emptyset\}$, the image by f^{deg} of $S \times \{\emptyset\}$ is equal to $T \times \{\emptyset\}$. Hence, let $\delta \in T$ be such that $f(\gamma', \emptyset)=(\delta, \emptyset)$. Since for all $\delta' \in T$, $f(\gamma', \emptyset) \neq (\delta', \emptyset)$, $f(\gamma', \emptyset) \neq (\delta, \emptyset)$: a contradiction. In the latter case, $T \neq \emptyset$. Since $f(\alpha, S)=(\beta, T)$, $S \neq \emptyset$. Thus, $(\alpha, S) \in \mathbf{MOD}_k \setminus \mathbf{MOD}_k^{\mathrm{deg}}$ and, E being the nonempty set of n-tuples of bits such that the image by f^{deg} of $S \times \{\emptyset\}$ is equal to $E \times \{\emptyset\}$, $f(\alpha, S)=f^E(\alpha, S)$. Since $f(\alpha, S)=(\beta, T)$, $f^E(\alpha, S)=(\beta, T)$. Consequently, $T=E$. Since the image by f^{deg} of $S \times \{\emptyset\}$ is equal to $E \times \{\emptyset\}$, the image by f^{deg} of $S \times \{\emptyset\}$ is equal to $T \times \{\emptyset\}$. Hence, let $\gamma \in S$ be such that $f(\gamma, \emptyset)=(\delta'', \emptyset)$. Since for all $\gamma'' \in S$, $f(\gamma'', \emptyset) \neq (\delta'', \emptyset)$, $f(\gamma, \emptyset) \neq (\delta'', \emptyset)$: a contradiction. □

Proof of Lemma 6: Let $(\alpha, S), (\beta, T) \in \mathbf{MOD}_k$. Suppose $f(\alpha, S)=f(\beta, T)$. We consider the following cases.

Case $S=\emptyset$. Since $f(\alpha, S)=f(\beta, T)$, by Lemma 5, $T=\emptyset$. Since $S=\emptyset$, $(\alpha, S), (\beta, T) \in \mathbf{MOD}_k^{\mathrm{deg}}$. Hence, $f(\alpha, S)=f^{\mathrm{deg}}(\alpha, S)$ and $f(\beta, T)=f^{\mathrm{deg}}(\beta, T)$. Since $f(\alpha, S)=f(\beta, T)$, $f^{\mathrm{deg}}(\alpha, S)=f^{\mathrm{deg}}(\beta, T)$. Thus, $(\alpha, S) \sim_k (\beta, T)$. Consequently, $g_{(k,\sigma)}(\alpha, S)=g_{(k,\sigma)}(\beta, T)$.

Case $S \neq \emptyset$. Since $f(\alpha, S)=f(\beta, T)$, by Lemma 5, $T \neq \emptyset$. Since $S \neq \emptyset$, $(\alpha, S), (\beta, T) \in \mathbf{MOD}_k \setminus \mathbf{MOD}_k^{\mathrm{deg}}$. Hence, $f(\alpha, S)=f^E(\alpha, S)$ and $f(\beta, T)=f^F(\beta, T)$, E being the nonempty set of n-tuples of bits such that the image by f^{deg} of $S \times \{\emptyset\}$ is equal to $E \times \{\emptyset\}$ and F being the nonempty set of n-tuples of bits such that the image by f^{deg} of $T \times \{\emptyset\}$ is equal to $F \times \{\emptyset\}$. Since $f(\alpha, S)=f(\beta, T)$, $f^E(\alpha, S)=f^F(\beta, T)$. Thus, $E=F$. Since $f^E(\alpha, S)=f^F(\beta, T)$, $(\alpha, S) \sim_k (\beta, T)$. Consequently, $g_{(k,\sigma)}(\alpha, S)=g_{(k,\sigma)}(\beta, T)$. □

Proof of Lemma 7: By induction on ψ. Let $(\beta, T) \in \mathbf{MOD}_n$. We consider the following cases.

Case $\psi=x_i$ for some $i \in \{1, \ldots, n\}$. (i)⇒(ii) Suppose $f(\alpha, S)=(\beta, T)$ and $(\alpha, S) \models_k \sigma(x_i)$ for some $(\alpha, S) \in \mathbf{MOD}_k$. Let $(\alpha', S') \in \mathbf{MOD}_k$. Suppose $f(\alpha', S')=(\beta, T)$. Since $f(\alpha, S)=(\beta, T)$, $f(\alpha, S)=f(\alpha', S')$. Hence, $g_{(k,\sigma)}(\alpha, S)=g_{(k,\sigma)}(\alpha', S')$. Thus, by Proposition 9, $(\alpha, S) \models_k \sigma(x_i)$ iff $(\alpha', S') \models_k \sigma(x_i)$. Since $(\alpha, S) \models_k \sigma(x_i)$, $(\alpha', S') \models_k \sigma(x_i)$.

(ii)⇒(iii) Suppose for all $(\alpha, S) \in \mathbf{MOD}_k$, if $f(\alpha, S)=(\beta, T)$ then $(\alpha, S) \models_k \sigma(x_i)$. Since f is surjective, let $(\alpha', S') \in \mathbf{MOD}_k$ be such that $f(\alpha', S')=(\beta, T)$. Since for all $(\alpha, S) \in \mathbf{MOD}_k$, if $f(\alpha, S)=(\beta, T)$ then $(\alpha, S) \models_k \sigma(x_i)$, $(\alpha', S') \models_k \sigma(x_i)$. Consequently, $\mathbf{for}_n(f(\alpha', S'))$ is one of the disjuncts of $\tau(x_i)$. Since $f(\alpha', S')=(\beta, T)$, by Proposition 5, $(\beta, T) \models_n \tau(x_i)$.

(iii)⇒(i) Suppose $(\beta, T) \models_n \tau(x_i)$. Hence, let $(\alpha, S) \in \mathbf{MOD}_k$ be such that $(\alpha, S) \models_k \sigma(x_i)$ and $(\beta, T) \models_n \mathbf{for}_n(f(\alpha, S))$. Thus, by Proposition 5, $f(\alpha, S)=(\beta, T)$.

Case $\psi=\bot$. (i)⇒(ii) Obviously, the condition "$f(\alpha, S)=(\beta, T)$ and $(\alpha, S) \models_k \sigma(\bot)$ for some $(\alpha, S) \in \mathbf{MOD}_k$ cannot hold.

(ii)⇒(iii) Since f is surjective, the condition "for all $(\alpha, S) \in \mathbf{MOD}_k$, if $f(\alpha, S)=(\beta, T)$ then $(\alpha, S) \models_k \sigma(\bot)$" cannot hold.

(iii)⇒(i) Obviously, the condition "$(\beta, T) \models_n \tau(\bot)$" cannot hold.

Case $\psi=\neg\chi$. (i)⇒(ii) Suppose $f(\alpha, S)=(\beta, T)$ and $(\alpha, S) \models_k \sigma(\neg\chi)$ for some $(\alpha, S) \in \mathbf{MOD}_k$. Let $(\alpha', S') \in \mathbf{MOD}_k$. Suppose $f(\alpha', S')=(\beta, T)$. For the sake of the contradiction, suppose $(\alpha', S') \not\models_k \sigma(\neg\chi)$. Consequently, $(\alpha', S') \models_k \sigma(\chi)$. Since $f(\alpha, S)=(\beta, T)$ and $f(\alpha', S')=(\beta, T)$, by induction hypothesis, $(\alpha, S) \models_k \sigma(\chi)$. Hence, $(\alpha, S) \not\models_k \sigma(\neg\chi)$: a contradiction.

(ii)⇒(iii) Suppose for all $(\alpha, S)\in\mathbf{MOD}_k$, if $f(\alpha, S)=(\beta, T)$ then $(\alpha, S)\models_k\sigma(\neg\chi)$. For the sake of the contradiction, suppose $(\beta, T)\not\models_n\tau(\neg\chi)$. Thus, $(\beta, T)\models_n\tau(\chi)$. Since f is surjective, let $(\alpha', S')\in\mathbf{MOD}_k$ be such that $f(\alpha', S')=(\beta, T)$. Since for all $(\alpha, S)\in\mathbf{MOD}_k$, if $f(\alpha, S)=(\beta, T)$ then $(\alpha, S)\models_k\sigma(\neg\chi)$, $(\alpha', S')\models_k\sigma(\neg\chi)$. Consequently, $(\alpha', S')\not\models_k\sigma(\chi)$. Since $f(\alpha', S')=(\beta, T)$, by induction hypothesis, $(\beta, T)\not\models_n\tau(\chi)$: a contradiction.

(iii)⇒(i) Suppose $(\beta, T)\models_n\tau(\neg\chi)$. Since f is surjective, let $(\alpha, S)\in\mathbf{MOD}_k$ be such that $f(\alpha, S)=(\beta, T)$. For the sake of the contradiction, suppose $(\alpha, S)\not\models_k\sigma(\neg\chi)$. Hence, $(\alpha, S)\models_k\sigma(\chi)$. Since $f(\alpha, S)=(\beta, T)$, by induction hypothesis, $(\beta, T)\models_n\tau(\chi)$. Thus, $(\beta, T)\not\models_n\tau(\neg\chi)$: a contradiction.

Case $\psi=\chi\vee\theta$. (i)⇒(ii) Suppose $f(\alpha, S)=(\beta, T)$ and $(\alpha, S)\models_k\sigma(\chi\vee\theta)$ for some $(\alpha, S)\in\mathbf{MOD}_k$. Let $(\alpha', S')\in\mathbf{MOD}_k$. Suppose $f(\alpha', S')=(\beta, T)$. For the sake of the contradiction, suppose $(\alpha', S')\not\models_k\sigma(\chi\vee\theta)$. Consequently, neither $(\alpha', S')\models_k\sigma(\chi)$, nor $(\alpha', S')\models_k\sigma(\theta)$. Since $f(\alpha, S)=(\beta, T)$ and $f(\alpha', S')=(\beta, T)$, by induction hypothesis, neither $(\alpha, S)\models_k\sigma(\chi)$, nor $(\alpha, S)\models_k\sigma(\theta)$. Hence, $(\alpha, S)\not\models_k\sigma(\chi\vee\theta)$: a contradiction.

(ii)⇒(iii) Suppose for all $(\alpha, S)\in\mathbf{MOD}_k$, if $f(\alpha, S)=(\beta, T)$ then $(\alpha, S)\models_k\sigma(\chi\vee\theta)$. For the sake of the contradiction, suppose $(\beta, T)\not\models_n\tau(\chi\vee\theta)$. Thus, neither $(\beta, T)\models_n\tau(\chi)$, nor $(\beta, T)\models_n\tau(\theta)$. Since f is surjective, let $(\alpha', S')\in\mathbf{MOD}_k$ be such that $f(\alpha', S')=(\beta, T)$. Since for all $(\alpha, S)\in\mathbf{MOD}_k$, if $f(\alpha, S)=(\beta, T)$ then $(\alpha, S)\models_k\sigma(\chi\vee\theta)$, $(\alpha', S')\models_k\sigma(\chi\vee\theta)$. Consequently, either $(\alpha', S')\models_k\sigma(\chi)$, or $(\alpha', S')\models_k\sigma(\theta)$. Since $f(\alpha', S')=(\beta, T)$, by induction hypothesis, either $(\beta, T)\models_n\tau(\chi)$, or $(\beta, T)\models_n\tau(\theta)$: a contradiction.

(iii)⇒(i) Suppose $(\beta, T)\models_n\tau(\chi\vee\theta)$. Since f is surjective, let $(\alpha, S)\in\mathbf{MOD}_k$ be such that $f(\alpha, S)=(\beta, T)$. For the sake of the contradiction, suppose $(\alpha, S)\not\models_k\sigma(\chi\vee\theta)$. Hence, neither $(\alpha, S)\models_k\sigma(\chi)$, nor $(\alpha, S)\models_k\sigma(\theta)$. Since $f(\alpha, S)=(\beta, T)$, by induction hypothesis, neither $(\beta, T)\models_n\tau(\chi)$, nor $(\beta, T)\models_n\tau(\theta)$. Thus, $(\beta, T)\not\models_n\tau(\chi\vee\theta)$: a contradiction.

Case $\psi=\Box\chi$. (i)⇒(ii) Suppose there exists $(\alpha, S)\in\mathbf{MOD}_k$ such that $f(\alpha, S)=(\beta, T)$ and $(\alpha, S)\models_k\sigma(\Box\chi)$. Let $(\alpha', S')\in\mathbf{MOD}_k$. Suppose $f(\alpha', S')=(\beta, T)$. For the sake of the contradiction, suppose $(\alpha', S')\not\models_k\sigma(\Box\chi)$. Consequently, let $\gamma'\in S'$ be such that $(\gamma', \emptyset)\not\models_k\sigma(\chi)$. Since f is a (k, n)-morphism and $f(\alpha', S')=(\beta, T)$, let $\delta\in T$ be such that $f(\gamma', \emptyset)=(\delta, \emptyset)$. Since f is a (k, n)-morphism and $f(\alpha, S)=(\beta, T)$, let $\gamma\in S$ be such that $f(\gamma, \emptyset)=(\delta, \emptyset)$. Since $(\gamma', \emptyset)\not\models_k\sigma(\chi)$ and $f(\gamma', \emptyset)=(\delta, \emptyset)$, by induction hypothesis, $(\gamma, \emptyset)\not\models_k\sigma(\chi)$. Hence, $(\alpha, S)\not\models_k\sigma(\Box\chi)$: a contradiction.

(ii)⇒(iii) Suppose for all $(\alpha, S)\in\mathbf{MOD}_k$, if $f(\alpha, S)=(\beta, T)$ then $(\alpha, S)\models_k\sigma(\Box\chi)$. For the sake of the contradiction, suppose $(\beta, T)\not\models_n\tau(\Box\chi)$. Thus, let $\delta\in T$ be such that $(\delta, \emptyset)\not\models_n\tau(\chi)$. Since f is surjective, let $(\alpha', S')\in\mathbf{MOD}_k$ be such that $f(\alpha', S')=(\beta, T)$. Since for all $(\alpha, S)\in\mathbf{MOD}_k$, if $f(\alpha, S)=(\beta, T)$ then $(\alpha, S)\models_k\sigma(\Box\chi)$, $(\alpha', S')\models_k\sigma(\Box\chi)$. Since f is a (k, n)-morphism and $f(\alpha', S')=(\beta, T)$, let $\gamma'\in S'$ be such that $f(\gamma', \emptyset)=(\delta, \emptyset)$. Since $(\delta, \emptyset)\not\models_n\tau(\chi)$, by induction hypothesis, $(\gamma', \emptyset)\not\models_k\sigma(\chi)$. Consequently, $(\alpha', S')\not\models_k\sigma(\Box\chi)$: a contradiction.

(iii)⇒(i) Suppose $(\beta, T)\models_n\tau(\Box\chi)$. Since f is surjective, let $(\alpha, S)\in\mathbf{MOD}_k$ be such that $f(\alpha, S)=(\beta, T)$. For the sake of the contradiction, suppose $(\alpha, S)\not\models_k\sigma(\Box\chi)$. Hence, let $\gamma\in S$ be such that $(\gamma, \emptyset)\not\models_k\sigma(\chi)$. Since f is a (k, n)-morphism and $f(\alpha, S)=(\beta, T)$, let $\delta\in T$ be such that $f(\gamma, \emptyset)=(\delta, \emptyset)$. Since $(\gamma, \emptyset)\not\models_k\sigma(\chi)$, by induction hypothesis, $(\delta, \emptyset)\not\models_n\tau(\chi)$. Thus, $(\beta, T)\not\models_n\tau(\Box\chi)$: a contradiction. □

Proof of Lemma 8: Let $(\beta, T)\in\mathbf{MOD}_k$ and all $i\in\{1, \ldots, n\}$. For the sake of the contradiction, suppose either $(\beta, T)\models_k\nu(x_i)$ and $f(\beta, T)\not\models_n x_i$, or $(\beta, T)\not\models_k\nu(x_i)$ and $f(\beta, T)\models_n x_i$. In the former case, by definition of ν, let $(\alpha, S)\in\mathbf{MOD}_k$ be such that $f(\alpha, S)\models_n x_i$ and $(\beta, T)\models_k\mathbf{for}_k(\alpha, S)$. Hence, by Proposition 5, $(\beta, T)=(\alpha, S)$. Since $f(\alpha, S)\models_n x_i$,

$f(\beta, T) \models_n x_i$: a contradiction. In the latter case, by definition of ν, $\mathbf{for}_k(\beta, T)$ is one of the disjuncts of $\nu(x_i)$. Since by Proposition 5, $(\beta, T) \models \mathbf{for}_k(\beta, T)$, $(\beta, T) \models_k \nu(x_i)$: a contradiction. □

Proof of Lemma 9: Let $(\beta, T) \in \mathbf{MOD}_k$ and $(\gamma, U) \in \mathbf{MOD}_n$. For the sake of the contradiction, suppose either $f(\beta, T) = (\gamma, U)$ and $(\beta, T) \not\models_k \nu(\mathbf{for}_n(\gamma, U))$, or $f(\beta, T) \neq (\gamma, U)$ and $(\beta, T) \models_k \nu(\mathbf{for}_n(\gamma, U))$. In the former case, since by Proposition 4, $(\gamma, U) \models_n \bar{x}^\gamma$, $f(\beta, T) \models_n \bar{x}^\gamma$. Hence, by Lemma 8, $(\beta, T) \models_k \nu(\bar{x}^\gamma)$. Since f is a (k, n)-morphism and by Proposition 4, for all $\gamma''', \gamma'''' \in U$, $(\gamma''', \emptyset) \models_n \bar{x}^{\gamma''''}$ iff $\gamma''' = \gamma''''$, for all $\beta' \in T$, there exists $\gamma' \in U$ such that $f(\beta', \emptyset) \models_n \bar{x}^{\gamma'}$ and for all $\gamma'' \in U$, there exists $\beta'' \in T$ such that $f(\beta'', \emptyset) \models_n \bar{x}^{\gamma''}$. Thus, by Lemma 8, for all $\beta' \in T$, there exists $\gamma' \in U$ such that $(\beta', \emptyset) \models_k \nu(\bar{x}^{\gamma'})$ and for all $\gamma'' \in U$, there exists $\beta'' \in T$ such that $(\beta'', \emptyset) \models_k \nu(\bar{x}'')$. Since $(\beta, T) \models_k \nu(\bar{x}^\gamma)$, $(\beta, T) \models_k \nu(\mathbf{for}_n(\gamma, U))$: a contradiction. In the latter case, $(\beta, T) \models_k \nu(\bar{x}^\gamma)$. Moreover, for all $\beta' \in T$, there exists $\gamma' \in U$ such that $(\beta', \emptyset) \models_k \nu(\bar{x}^{\gamma'})$ and for all $\gamma'' \in U$, there exists $\beta'' \in T$ such that $(\beta'', \emptyset) \models_k \nu(\bar{x}'')$. Consequently, by Lemma 8, for all $\beta' \in T$, there exists $\gamma' \in U$ such that $f(\beta', \emptyset) \models_n \bar{x}^{\gamma'}$ and for all $\gamma'' \in U$, there exists $\beta'' \in T$ such that $f(\beta'', \emptyset) \models_n \bar{x}^{\gamma''}$. Since f is a (k, n)-morphism and by Proposition 4, for all $\gamma''', \gamma'''' \in U$, $(\gamma''', \emptyset) \models_n \bar{x}^{\gamma''''}$ iff $\gamma''' = \gamma''''$, for all $\beta' \in T$, there exists $\gamma' \in U$ such that $f(\beta', \emptyset) = (\gamma', \emptyset)$ and for all $\gamma'' \in U$, there exists $\beta'' \in T$ such that $f(\beta'', \emptyset) = (\gamma'', \emptyset)$. Since $(\beta, T) \models_k \nu(\bar{x}^\gamma)$, by Proposition 11, $f(\beta, T) = (\gamma, U)$: a contradiction. □

Proof of Lemma 10: Let $(\beta, T) \in \mathbf{MOD}_k$ and $i \in \{1, \dots, n\}$. For the sake of the contradiction, suppose either $(\beta, T) \models_k \nu(\tau(x_i))$ and $(\beta, T) \not\models_k \sigma(x_i)$, or $(\beta, T) \not\models_k \nu(\tau(x_i))$ and $(\beta, T) \models_k \sigma(x_i)$. In the former case, by definition of τ, let $(\alpha, S) \in \mathbf{MOD}_k$ be such that $(\alpha, S) \models_k \sigma(x_i)$ and $(\beta, T) \models_k \nu(\mathbf{for}_n(f(\alpha, S)))$. Hence, by Lemma 9, $f(\beta, T) = f(\alpha, S)$. Thus, $g_{(k,\sigma)}(\beta, T) = g_{(k,\sigma)}(\alpha, S)$. Consequently, by Proposition 9, $(\beta, T) \models_k \sigma(x_i)$ iff $(\alpha, S) \models_k \sigma(x_i)$. Since $(\alpha, S) \models_k \sigma(x_i)$, $(\beta, T) \models_k \sigma(x_i)$: a contradiction. In the latter case, by definition of τ, $\mathbf{for}_n(f(\beta, T))$ is one of the disjuncts of $\tau(x_i)$. Since by Lemma 9, $(\beta, T) \models_k \nu(\mathbf{for}_n f(\beta, T))$, $(\beta, T) \models_k \nu(\tau(x_i))$: a contradiction. □

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