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The discriminant criterion and automorphism groups of quantized algebras

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ABSTRACT

We compute the automorphism groups of some quantized algebras, including tensor products of quantum Weyl algebras and some skew polynomial rings.

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0. Introduction

It is well known that every automorphism of the polynomial ring $k[x]$, where k is a field, is determined by the assignment $x \mapsto ax + b$ for some $a \in k^\times := k \setminus \{0\}$ and $b \in k$. Every automorphism of $k[x_1, x_2]$ is *tame*, that is, it is generated by affine and elementary automorphisms (defined below). This result was first proved by Jung [10] in 1942 for characteristic zero and then by van der Kulk [20] in 1953 for arbitrary characteristic. A structure theorem for the automorphism group of $k[x_1, x_2]$ was also given in [20]. The automorphism group of $k[x_1, x_2, x_3]$ has not yet been fully understood, and the best result in this direction is the existence of wild automorphisms (e.g. the Nagata automorphism) by Shestakov–Umirbaev [16].

The automorphism group of the skew polynomial ring $k_q[x_1, \dots, x_n]$, where $q \in k^\times$ is not a root of unity and $n \geq 2$, was completely described by Alev and Chamarie [2, Theorem 1.4.6] in 1992. Since then, many researchers have been successfully computing the automorphism groups of classes of interesting infinite-dimensional noncommutative algebras, including certain quantum groups, generalized quantum Weyl algebras, skew polynomial rings and many more – see [2–5,8,19,21,22], among others. In particular, Yakimov has proved the Andruskiewitsch–Dumas conjecture and the Launois–Lenagan conjecture by using a rigidity theorem for quantum tori, see [21,22], each of which determines the automorphism group of a family of quantized algebras with parameter q being not a root of unity. See also [9] for a uniform approach to these two conjectures.

Determining the automorphism group of an algebra is generally a very difficult problem. In [6] we introduced the discriminant method to compute automorphism groups of some noncommutative algebras. In this paper we continue to develop new methods and extend ideas from [6] for both discriminants and automorphism groups.

Suppose A is a filtered algebra with filtration $\{F_i A\}_{i \geq 0}$ such that the associated graded algebra $\operatorname{gr} A$ is generated in degree 1. An automorphism g of A is *affine* if $g(F_1 A) \subset F_1 A$. An automorphism h of the polynomial extension $A[t]$ is called *triangular* if there is a $g \in \operatorname{Aut}(A)$, $c \in k^\times$ and r in the center of A such that

$$h(t) = ct + r \quad \text{and} \quad h(x) = g(x) \in A \quad \text{for all } x \in A.$$

As in [6], we use the discriminant to control automorphisms and locally nilpotent derivations. Let $C(A)$ denote the center of A. Here is the discriminant criterion for affine automorphisms.

Theorem 1. *Assume k is a field of characteristic 0. Let A be a filtered algebra, finite over its center, such that the associated graded ring $\operatorname{gr} A$ is a connected graded domain. Suppose that the v-discriminant $d_v(A/C(A))$ is dominating for some $v \geq 1$. Then the following hold.*

(1) *Every automorphism of A is affine, and $\operatorname{Aut}(A)$ is an algebraic group that fits into the exact sequence*

$$1 \to (k^\times)^r \to \mathrm{Aut}(A) \to S \to 1, \qquad (*)$$

where $r \geq 0$ *and* S *is a finite group. If* S *can be realized as a subgroup of* $\mathrm{Aut}(A)$*, then* $\mathrm{Aut}(A) = S \ltimes (k^\times)^r$*.*

(2) *Every automorphism of the polynomial extension* $A[t]$ *is triangular.*

(3) *Every locally nilpotent derivation of* A *is zero.*

The terminology will be explained in Section 1. This is proved below (in slightly more general form) as Theorem 1.13.

The discriminant criterion is very effective in computing the automorphism group for a large class of noncommutative algebras (examples can be found in [6] and in this paper), but the computation of the discriminant can be difficult. It would be nice to develop new theories and efficient computational tools for the discriminant in the setting of noncommutative algebra.

In this paper we apply our methods to two families of quantized algebras: quantum Weyl algebras and skew polynomial rings. We recall these next.

Let q be a nonzero scalar in k and let A_q be the q*-quantum Weyl algebra*, the algebra generated by x and y subject to the relation $yx = qxy + 1$ (we assume that $q \neq 1$, but q need not be a root of unity). Consider the tensor product $B := A_{q_1} \otimes \cdots \otimes A_{q_m}$ of quantum Weyl algebras, where $q_i \in k^\times \setminus \{1\}$ for all i. Since we are not assuming that the q_i are roots of unity, B need not be finite over its center and so the hypotheses of Theorem 1 might fail; however, the conclusions hold.

Theorem 2. *Let* k *be a field. Let* $B = A_{q_1} \otimes \cdots \otimes A_{q_m}$ *and assume that* $q_i \neq 1$ *for all* $i = 1, \ldots, m$*. Then the following hold.*

(1) *Every automorphism of* B *is affine, and* $\mathrm{Aut}(B)$ *is an algebraic group that fits into an exact sequence of the form* (*)*, with* $r = m$*.*

(2) *The automorphism group of* $B[t]$ *is triangular.*

(3) *If* $\mathrm{char}\, k = 0$*, then every locally nilpotent derivation of* B *is zero.*

See Section 5 for the proof. As a consequence of Theorem 2, the following hold (Theorem 5.7):

- If $q_i \neq \pm 1$ and $q_i \neq q_j^{\pm 1}$ for all $i \neq j$, then $\mathrm{Aut}(B) = (k^\times)^m$.
- If $q_i = q \neq \pm 1$ for all i, then $\mathrm{Aut}(B) = S_m \ltimes (k^\times)^m$.

Let $\{p_{ij} \in k^\times \mid 1 \leq i < j \leq n\}$ be a set of parameters, and set $p_{ji} = p_{ij}^{-1}$ and $p_{ii} = p_{jj} = 1$ for all $i < j$. In this paper, a *skew polynomial ring* is defined to be the algebra generated by $x_1, \ldots, x_n$ subject to the relations $x_j x_i = p_{ij} x_i x_j$ for all $i < j$, and is denoted by $k_{p_{ij}}[x_1, \ldots, x_n]$. Recall from [14, Chapter 13] that a *PI algebra* is one which satisfies a *polynomial identity*. Skew polynomial rings are PI if and only if they are finite over their center; hence the skew polynomial ring $k_{p_{ij}}[x_1, \ldots, x_n]$ is PI if and only if each p_{ij} is a root of unity. The automorphism groups of skew polynomial rings have been studied by several authors [2,22]. The next result says that the discriminant criterion works well for PI skew polynomial rings.

Theorem 3 *(Theorem 3.1). Let $A = k_{p_{ij}}[x_1, \ldots, x_n]$ be a PI skew polynomial ring over the commutative domain k. Then the following are equivalent.*

(1) *$d_w(A/C(A))$ is dominating, where $w = \mathrm{rk}(A/C(A))$.*
(2) *Every automorphism of A is affine.*
(3) *Every automorphism of $A[t]$ is triangular.*
(4) *$C(A) \subset k\langle x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\rangle$ for some $\alpha_1, \ldots, \alpha_n \geq 2$.*

If $\mathbb{Z} \subset k$, then the above are also equivalent to

(5) *Every locally nilpotent derivation is zero.*

Note that the implication (1) ⇒ (5) fails when $\mathrm{char}\, k \neq 0$ [6, Example 3.9].

One example is $k_q[x_1, \ldots, x_n]$ with n even and $q \neq 1$ a primitive ℓth root of unity. In this case, $C(A) = k[x_1^\ell, \ldots, x_n^\ell]$, so part (4) of the above holds. Therefore all of (1)–(5) hold. By part (2), $\mathrm{Aut}(k_q[x_1, \ldots, x_n])$ is affine. An easy computation shows that

$$\mathrm{Aut}(k_q[x_1, \ldots, x_n]) = \begin{cases} (k^\times)^n & \text{if} \quad q \neq \pm 1, \\ S_n \ltimes (k^\times)^n & \text{if} \quad q = -1. \end{cases} \tag{0.3.1}$$

If n is odd and q is a root of unity, then $\mathrm{Aut}(k_q[x_1, \ldots, x_n])$ is not affine – see Example 1.8 – and is much more complicated. The structure of $\mathrm{Aut}(k_q[x_1, \ldots, x_n])$ is not well understood for n odd, even when $n = 3$.

We have some results concerning automorphisms of not necessarily PI skew polynomial rings. We need to introduce some notation. For any $1 \leq s \leq n$, let

$$T_s = \{(d_1, \ldots, \widehat{d_s}, \ldots, d_n) \in \mathbb{N}^{n-1} \mid \prod_{\substack{j=1 \\ j \neq s}}^{n} p_{ij}^{d_j} = p_{is} \ \forall\, i \neq s\}.$$

We show in Theorem 3.8 that in the PI case, if $T_s = \emptyset$ for all s, then every automorphism of A is affine. Note also that in the PI case, if T_s is nonempty, then T_s is in fact infinite. If we drop the PI assumption and we allow at most one T_s to be infinite, we can still understand the automorphism group, as described in the next result.

An automorphism g of $k_{p_{ij}}[x_1, \ldots, x_n]$ is called *elementary* if there is an s and an element f generated by $x_1, \ldots, \widehat{x_s}, \ldots, x_n$ such that

$$g(x_i) = \begin{cases} x_i & i \neq s \\ x_s + f & i = s. \end{cases}$$

An automorphism of $k_{p_{ij}}[x_1, \ldots, x_n]$ is called *tame* if it is generated by affine and elementary automorphisms.

Theorem 4. *Let $A = k_{p_{ij}}[x_1, \ldots, x_n]$ be a (not necessarily PI) skew polynomial algebra over the commutative domain k, and suppose that x_i is not central in A for all i. Let s_0 be some integer between 1 and n. Suppose that T_s is finite for all $s \neq s_0$. Then every automorphism of A is tame.*

This is proved as a consequence of Theorem 3.11.

The paper is laid out as follows. In Section 1, we introduce the notion of the discriminant and prove Theorem 1 – note that this result can be viewed as a generalization of [6, Theorem 3]. In Section 2, we compute the discriminants of skew polynomial rings over their center. In Section 3, we prove that $\mathrm{Aut}(k_{p_{ij}}[x_1, \ldots, x_n])$ is affine if and only if the discriminant is dominating and then prove Theorems 3 and 4. We discuss some properties of automorphisms and discriminants in Section 4. In the final section, we prove Theorem 2.

1. The discriminant controls automorphisms

Throughout the rest of the paper let k be a commutative domain, and sometimes we further assume that k is a field. Modules, vector spaces, algebras, tensor products, and morphisms are over k. All algebras are associative with unit.

The beginning of this section overlaps with the paper [6]. We start by recalling the concept of the discriminant in the noncommutative setting. Let R be a commutative algebra and let B and F be algebras both of which contain R as a subalgebra. In applications, F would be either R or a ring of fractions of R. An R-linear map $\mathrm{tr} : B \to F$ is called a *trace map* if $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in B$.

If B is the $w \times w$-matrix algebra $M_w(R)$ over R, we have the internal trace $\mathrm{tr}_{\mathrm{int}} : B \to R$ defined to be the usual matrix trace, namely, $\mathrm{tr}_{\mathrm{int}}((r_{ij})) = \sum_{i=1}^{w} r_{ii}$. Let B be an R-algebra, let F be a localization of R, and suppose that $B_F := B \otimes_R F$ is finitely generated free over F. Then left multiplication defines a natural embedding of R-algebras $lm : B \to B_F \to \mathrm{End}_F(B_F) \cong M_w(F)$, where w is the rank $\mathrm{rk}(B_F/F)$. Then we define the *regular trace map* by composing:

$$\mathrm{tr}_{\mathrm{reg}} : B \xrightarrow{lm} M_w(F) \xrightarrow{\mathrm{tr}_{\mathrm{int}}} F.$$

Usually we use the regular trace even if other trace maps exist. The following definition is well known; see Reiner's book [15]. Let $R^\times$ denote the set of invertible elements in R. If $f, g \in R$ and $f = cg$ for some $c \in R^\times$, then we write $f =_{R^\times} g$.

Definition 1.1. (See [6, Definition 1.3].) Let $\mathrm{tr} : B \to F$ be a trace map and v be a fixed integer. Let $Z := \{z_i\}_{i=1}^{v}$ be a subset of B.

(1) The *discriminant* of Z is defined to be

$$d_v(Z : \mathrm{tr}) = \det(\mathrm{tr}(z_i z_j))_{v\times v} \in F.$$

(2) (See [15, Section 10, p. 126].) The *v-discriminant ideal* (or *v-discriminant R-module*) $D_v(B : \mathrm{tr})$ is the R-submodule of F generated by the set of elements $d_v(Z : \mathrm{tr})$ for all $Z = \{z_i\}_{i=1}^{v} \subset B$.

(3) Suppose B is an R-algebra which is finitely generated free over R of rank w. In this case, we take $F = R$. The *discriminant* of B over R is defined to be

$$d(B/R) =_{R^\times} d_w(Z : \mathrm{tr}),$$

where Z is an R-basis of B. Note that $d(B/R)$ is well-defined up to a scalar in $R^\times$ [15, p. 66, Exer 4.13].

We refer to the books [1,15,18] for the classical definition of discriminant and its connection with the above definition.

To cover a larger class of algebras, in particular those that are not free over their center, we need a modified version of the discriminant. Let B be a domain. A normal element $x \in B$ *divides* $y \in B$ if $y = wx$ for some $w \in B$. If $\mathcal{D} := \{d_i\}_{i\in I}$ is a set of elements in B, a normal element $x \in B$ is called a *common divisor* of $\mathcal{D}$ if x divides d_i for all $i \in I$. We say a normal element $x \in B$ is the *greatest common divisor* or *gcd* of $\mathcal{D}$, denoted by $\gcd \mathcal{D}$, if

(1) x is a common divisor of $\mathcal{D}$, and
(2) any common divisor y of $\mathcal{D}$ divides x.

It follows from part (2) that the gcd of any subset $\mathcal{D} \subset B$ (if it exists) is unique up to a scalar in $B^\times$.

Note that the gcd in B may be different from the gcd in R, if both exist. For example, the gcd in R could be 1 while the gcd in B is non-trivial. By definition, the gcd in R is a divisor of the gcd in B. Of course, the gcd in B may be more difficult to compute since B is typically noncommutative.

Definition 1.2. Let $\mathrm{tr} : B \to R$ be a trace map and v a positive integer. Let $Z = \{z_i\}_{i=1}^{v}$ and $Z' = \{z'_i\}_{i=1}^{v}$ be v-element subsets of B.

(1) The *discriminant* of the pair (Z, Z') is defined to be

$$d_v(Z, Z' : \mathrm{tr}) = \det(\mathrm{tr}(z_i z'_j))_{v\times v} \in R.$$

(2) The *modified v-discriminant ideal* $MD_v(B : \mathrm{tr})$ is the ideal of R generated by the set of elements $d_v(Z, Z' : \mathrm{tr})$ for all $Z, Z' \subset B$.
(3) The *v-discriminant* $d_v(B/R)$ is defined to be the gcd in B of the elements $d_v(Z, Z' : \mathrm{tr})$ for all $Z, Z' \subset B$. Equivalently, the v-discriminant $d_v(B/R)$ is the gcd in B of the elements in $MD_v(B : \mathrm{tr})$.

If $d_v(B/R)$ exists, then the ideal $(d_v(B/R))$ of B generated by $d_v(B/R)$ is the smallest principal ideal of B which is generated by a normal element and contains $MD_v(B : \mathrm{tr})B$.

It is clear that $D_v(B : \mathrm{tr}) \subset MD_v(B : \mathrm{tr})$. Equality should hold under reasonable hypotheses. For example, if B is an R-algebra which is finitely generated free over R and if $w = \mathrm{rk}(B/R)$, then $MD_w(B : \mathrm{tr})$ equals $D_w(B : \mathrm{tr})$, both of which are generated by the single element $d(B/R)$. In this case it is also true that $d(B/R) =_{B^\times} d_w(B/R)$. This follows from (1.10.2), which states that if Z and Z' are two R-bases of B, then

$$d(B/R) =_{R^\times} d_w(Z, Z' : \mathrm{tr}).$$

If $v_1 < v_2$ and if $d_{v_1}(B/R)$ and $d_{v_2}(B/R)$ exist, then $d_{v_1}(B/R)$ divides $d_{v_2}(B/R)$, by Lemma 1.4(5), and if $v > \mathrm{rk}(B/R)$, then $d_v(B/R) = 0$ (Lemma 1.9(2)).

If B is not free as an R-module, then to use Definition 1.2, we let F be a localization of R, typically its field of fractions, we let $\mathrm{tr} : B \to F$ be the regular trace, and we assume that the image of tr is in R. (This happens frequently when R is the center of B – see Lemma 2.7(9), for example.)

In [6], we computed some discriminants. Here are some new examples.

Example 1.3. Let k be a commutative domain such that 2 is nonzero in k. In parts (2) and (3) we further assume that 3 is nonzero in k and that $\xi \in k$ is a primitive third root of unity. Some details in the computations are omitted.

(1) Let R be a commutative domain, $0 \neq x \in R$, and let $A = \begin{pmatrix} R & R \\ xR & R \end{pmatrix}$. Then the center of A is R and $Z := \{e_{11}, e_{12}, xe_{21}, e_{22}\}$ is an R-basis of A. By using the regular trace tr, we have

$$\mathrm{tr}(e_{11}) = 2, \quad \mathrm{tr}(e_{12}) = 0, \quad \mathrm{tr}(xe_{21}) = 0, \quad \mathrm{tr}(e_{22}) = 2.$$

Using these traces and the fact tr is R-linear, we have the matrix

$$(\mathrm{tr}(z_i z_j))_{4\times 4} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2x & 0 \\ 0 & 2x & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and the discriminant $d(A/R)$ is -2^4x^2.

(2) Let $B = k_{p_{ij}}[x_1, x_2, x_3]$, where $p_{12} = -1$, $p_{13} = \xi$ and $p_{23} = 1$. Then the center R is the polynomial ring generated by x_1^6, x_2^2 and x_3^3. The algebra B is a free R-module with basis

$$Z := \{x_1^{i_1} x_2^{i_2} x_3^{i_3} \mid 0 \le i_1 \le 5, 0 \le i_2 \le 1, 0 \le i_3 \le 2\}.$$

The rank of B over R is 36. One can check that the regular traces are

$$\operatorname{tr}(1) = 36, \quad \operatorname{tr}(f) = 0 \;\; \forall\, f \in Z \setminus \{1\}.$$

The discriminant $d(B/R)$ is $(x_1^5 x_2 x_3^2)^{36}$ (Proposition 2.8).

(3) Let $C = k_{p_{ij}}[x_1, x_2, x_3]$, where $p_{12} = -1$, $p_{13} = -1$, and $p_{23} = 1$. Then the center R is generated by x_1^2, x_2^2, x_3^2, and $x_2 x_3$. So R is not a polynomial ring and C is not free over R. The rank of C over R is 4 and C is generated by the set $\{1, x_1, x_2, x_3, x_1x_2, x_1x_3\}$ over R. If F is the field of fractions of R, then one can show that the image of the regular trace $\operatorname{tr} : C \to F$ is in R. By a degree argument, the regular traces are

$$\operatorname{tr}(1) = 4, \quad \operatorname{tr}(x_1) = \operatorname{tr}(x_2) = \operatorname{tr}(x_3) = \operatorname{tr}(x_1x_2) = \operatorname{tr}(x_1x_3) = 0.$$

Since C is not free over R, we compute the modified discriminant ideal. A non-trivial computation shows that $MD_4(C : \operatorname{tr})$ is the ideal generated by $x_1^4 x_2^i x_3^{4-i}$ for $i = 0, 1, 2, 3, 4$ and $d_4(C/R) =_{k^\times} x_1^4$. In this example, it is also possible to compute $d_v(C/R)$ for other v:

$$d_v(C/R) =_{k^\times} \begin{cases} 0 & v > 4, \\ x_1^2 & v = 3, \\ 1 & v < 3. \end{cases}$$

(4) Let $D = k_{p_{ij}}[x_1, x_2, x_3]$, where $p_{12} = -1$, $p_{13} = -1$, and $p_{23} = i$ where $i^2 = -1$. Then the center R is generated by x_1^2, x_2^4, x_3^4, and $x_1 x_2^2 x_3^2$. As in the last example, R is not a polynomial ring and D is not free over R, but the image of the regular trace is in R. The rank of D over R is 16 and D is generated by $x_1^i x_2^j x_3^k$, where $0 \le i \le 1$, $0 \le j, k \le 3$ and $(i, j, k) \ne (1, 2, 2)$. One can check that

$$\operatorname{tr}(1) = 16, \quad \operatorname{tr}(x_1^i x_2^j x_3^k) = 0$$

for all $0 \le i \le 1$, $0 \le j, k \le 3$, and $(i, j, k) \ne (1, 2, 2)$ (Lemma 2.7(8)). The modified discriminant ideal $MD_{16}(D : \operatorname{tr})$ is generated by $x_2^{48} x_3^{48} \cdot f$, where f ranges over elements of the form $(x_1 x_2^{-2} x_3^2)^{i_1} (x_1 x_2^2 x_3^{-2})^{i_2} (x_1 x_2^{-2} x_3^{-2})^{i_3}$ for all $0 \le i_1, i_2, i_3 \le 8$. As a consequence, $d_{16}(D/R) =_{k^\times} x_2^{16} x_3^{16}$ (Lemma 1.11(4)).

(5) Let $E = k_{p_{ij}}[x_1, x_2, x_3]$, where $p_{12} = -1$, $p_{13} = \xi$, and $p_{23} = -1$. Then the center R is generated by x_1^6, x_2^2, x_3^6, and $x_1^3 x_3^3$, which is not a polynomial ring. The modified

discriminant ideal $MD_{36}(D : \mathrm{tr})$ is generated by

$$(x_1^2x_2x_3^2)^{36}x_1^{3i}x_3^{3(36-i)} \quad \text{for} \quad 0 \le i \le 36,$$

and so $d_{36}(E/R) =_{k^\times} (x_1^2x_2x_3^2)^{36}$ (Lemma 1.11(4)).

One of our key lemmas is the following, which suggests that the discriminant controls automorphisms.

Lemma 1.4. *Retain the notation as in Definitions 1.1 and 1.2. Suppose that* tr *is the regular trace and that the image of* tr *is in R. Let g be an automorphism of B such that g and g^{-1} preserve R.*

(1) *(See [6, Lemma 1.8(5)].) The discriminant ideal $D_w(B : \mathrm{tr})$ is g-invariant, where $w = \mathrm{rk}(B/R)$.*
(2) *(See [6, Lemma 1.8(6)].) If B is a finitely generated free module over R, then the discriminant $d(B/R)$ is g-invariant up to a unit of R.*
(3) *The modified discriminant ideal $MD_v(B : \mathrm{tr})$ is g-invariant for all v.*
(4) *The v-discriminant $d_v(B/R)$ is g-invariant up to a unit in B, for all v.*
(5) *For integers $v_1 < v_2$, $MD_{v_2}(B : \mathrm{tr}) \subset MD_{v_1}(B : \mathrm{tr})$. So if $d_{v_1}(B/R)$ and $d_{v_2}(B/R)$ exist, then $d_{v_1}(B/R)$ divides $d_{v_2}(B/R)$. As a consequence, the quotient $d_{v_2}(B/R)/d_{v_1}(B/R)$ is g-invariant up to a unit in B.*

Proof. (3) By [6, Lemma 1.8(2)], $\mathrm{tr}(g(x)) = g(\mathrm{tr}(x))$ for all $x \in B$. This implies that $g(d_v(Z, Z' : \mathrm{tr})) = d_v(g(Z), g(Z') : \mathrm{tr})$ for any $Z, Z' \subset B$. Therefore $g(MD_v(B : \mathrm{tr})) \subset MD_v(B : \mathrm{tr})$. Similarly, $g^{-1}(MD_v(B : \mathrm{tr})) \subset MD_v(B : \mathrm{tr})$. These imply that $g(MD_v(B : \mathrm{tr})) = MD_v(B : \mathrm{tr})$. The proof of (4) is similar.

(5) Let Z and Z' be any v_2-element subsets of B as in Definition 1.2. Use X for any v_1-element subset of Z and Y for $Z \setminus X$. We similarly define X' and Y'. By linear algebra,

$$\begin{aligned} d_{v_2}(Z, Z' : \mathrm{tr}) &= \det(\mathrm{tr}(z_i z_j'))_{v_2 \times v_2} \\ &= \sum_{X \subset Z, X' \subset Z'} \pm \det(\mathrm{tr}(x_i x_j'))_{v_1 \times v_1} \det(\mathrm{tr}(y_i y_j'))_{(v_2 - v_1) \times (v_2 - v_1)} \\ &= \sum_{X \subset Z, X' \subset Z'} \pm d_{v_1}(X, X' : \mathrm{tr}) d_{v_2 - v_1}(Y, Y' : \mathrm{tr}), \end{aligned}$$

which is in $MD_{v_1}(B : \mathrm{tr})$. Hence $MD_{v_2}(B : \mathrm{tr}) \subset MD_{v_1}(B : \mathrm{tr})$ and the second assertion follows. The consequence is clear. $\square$

The next proposition says that the discriminant controls locally nilpotent derivations. Recall that a k-linear map $\partial : B \to B$ is called a *derivation* if the Leibniz rule

$$\partial(xy) = \partial(x)y + x\partial(y)$$

holds for all $x, y \in B$. We call ∂ *locally nilpotent* if for every $x \in B$, $\partial^n(x) = 0$ for some n. Given a locally nilpotent derivation ∂ (and assuming that $\mathbb{Q} \subset k$), the exponential map $\exp(\partial) : B \to B$ is defined by

$$\exp(\partial)(x) = \sum_{i=0}^{\infty} \frac{1}{i!} \partial^i(x) \quad \text{for all } x \in B.$$

Since ∂ is locally nilpotent, $\exp(\partial)$ is an algebra automorphism of B with inverse $\exp(-\partial)$.

Proposition 1.5. *Assume that $\mathbb{Q} \subset k$ and that $B^\times = k^\times$. Let R be the center of B. Suppose that* tr *is the regular trace and that the image of* tr *is in R, and suppose that $d_v(B/R)$ exists. If ∂ is a locally nilpotent derivation of B, then $\partial(d_v(B/R)) = 0$. Similarly, if B is finitely generated free over R, then $\partial(d(B/R)) = 0$.*

Proof. For any $c \in k$, consider the algebra automorphism

$$\exp(c\partial) : x \longmapsto \sum_{i=0}^{\infty} \frac{c^i}{i!} \partial^i(x) \quad \text{for all } x \in B.$$

Let $x = d_v(B/R)$ (or $d(B/R)$ in the second case). Then, by Lemma 1.4(4), $\exp(c\partial)(x) = \lambda_c x \in kx$ for some $\lambda_c \in k^\times$. This is true for all $c \in \mathbb{Q}$. Since ∂ is locally nilpotent, there are only finitely many nonzero $\partial^i(x)$ terms for $i = 0, 1, 2, \ldots$. By using the Vandermonde determinant, $\partial^i(x) \in kx$ for all i. If $\partial(x) = ax$, then $\partial^i(x) = a^i x$ for all i. Since ∂ is locally nilpotent, $a = 0$ and $\partial(x) = 0$. $\square$

This proposition fails when k has positive characteristic [6, Example 3.9].

Let $C = \bigoplus_i C_i$ be a graded algebra over k. We say C is *connected graded* if $C_i = 0$ for $i < 0$ and $C_0 = k$, and C is *locally finite* if each C_i is finitely generated over k. We now consider filtered rings A. Let Y be a finitely generated free k-submodule of A such that $k \cap Y = \{0\}$. Consider the *standard filtration* defined by $F_n A := (k + Y)^n$ for all $n \geq 0$. Assume that this filtration is exhaustive and that the associated graded ring $\operatorname{gr} A$ is connected graded. For each element $f \in F_n A \setminus F_{n-1} A$, the associated element in $\operatorname{gr} A$ is defined to be $\operatorname{gr} f = f + F_{n-1} A \in (\operatorname{gr}_F A)_n$. The degree of a nonzero element $f \in A$, denoted by $\deg f$, is defined to be the degree of $\operatorname{gr} f$.

Suppose now A is generated by $Y = \bigoplus_{i=1}^n kx_i$, so with the standard filtration, the nonzero elements of Y have degree 1. A monomial $x_1^{b_1} \cdots x_n^{b_n}$ is said to have degree *component-wise less than* (or, *cwlt*, for short) $x_1^{a_1} \cdots x_n^{a_n}$ if $b_i \leq a_i$ for all i and $b_{i_0} < a_{i_0}$ for some i_0. We write $f = cx_1^{b_1} \cdots x_n^{b_n} + (\text{cwlt})$ if $f - cx_1^{b_1} \cdots x_n^{b_n}$ is a linear combination of monomials with degree component-wise less than $x_1^{b_1} \cdots x_n^{b_n}$.

Definition 1.6. Retain the above notation. Suppose that $Y = \bigoplus_{i=1}^n kx_i$ generates A as an algebra.

(1) A nonzero element $f \in A$ is called *locally* $(-s)$-*dominating* if, up to a permutation, f can be written as $f(x_1, x_2, \ldots, x_{n-s})$ such that, for every $g \in \mathrm{Aut}(A)$, one has
 (a) $\deg f(y_1, \ldots, y_{n-s}) \geq \deg f$, where $y_i = g(x_i)$ for all $i \leq n-s$, and
 (b) $\deg f(y_1, \ldots, y_{n-s}) > \deg f$ if, further, $\deg y_{i_0} > 1$ for some $i_0 \leq n-s$.

(2) Suppose $\mathrm{gr}\, A$ is a connected graded domain. A nonzero element $f \in A$ generated by $\{x_1, \ldots, x_{n-s}\}$ (up to a permutation of $\{x_i\}_{i=1}^n$) is called $(-s)$-*dominating* if, for every $\mathbb{N}$-filtered PI algebra T with $\mathrm{gr}\, T$ a connected graded domain, and for every subset $\{y_1, \ldots, y_{n-s}\} \subset T$ that is linearly independent in the quotient k-module T/F_0T, there is a lift of f, say $f(x_1, \ldots, x_{n-s})$, in the free algebra $k\langle x_1, \ldots, x_{n-s}\rangle$, such that the following hold: either $f(y_1, \ldots, y_{n-s}) = 0$ or
 (a) $\deg f(y_1, \ldots, y_{n-s}) \geq \deg f$, and
 (b) $\deg f(y_1, \ldots, y_{n-s}) > \deg f$ if, further, $\deg y_{i_0} > 1$ for some $i_0 \leq n-s$.

If $f = x_1^{b_1} \cdots x_{n-s}^{b_{n-s}} + (\mathrm{cwlt})$ for some $b_1, \ldots, b_{n-s} \geq 1$, then f is $(-s)$-dominating: see the proof of [6, Lemma 2.2]. It is easy to check that $(-s)$-dominating elements are indeed locally $(-s)$-dominating.

Note that the notation of “0-dominating” is exactly the notation of “dominating” of [6, Definition 2.1(2)] and the notation of “locally 0-dominating” is exactly the notation of “locally dominating” of [6, Definition 2.1(1)].

Definition 1.7. Let (A, Y) be defined as above. In particular, $Y = \bigoplus_{i=1}^n kx_i$ generates A as an algebra.

(1) An algebra automorphism g of A is said to be $(-s)$-*affine* if $\deg g(x_i) = 1$ for all but s-many values of i. A 0-affine automorphism is also called an *affine* automorphism [6, Definition 2.4(1)].

(2) Let C be an algebra over k. A k-algebra automorphism g of $A \otimes C$ is said to be $(-s)$-C-*affine* if $g(x_i) \subset (Y \oplus k) \otimes C$ for all but s-many values of i. A 0-C-affine automorphism is also called a C-*affine* automorphism.

The definition of $(-s)$-affine depends on the choice of generators – one can construct an example showing this just by a linear change of variables. This is unfortunate, but this definition does work well for a natural choice of generators for skew polynomial algebras.

Note that any elementary automorphism is (-1)-affine. The next example shows that not every automorphism is affine.

Example 1.8. For $q \in k^\times$, let $k_q[x_1, \ldots, x_n]$ be the q-skew polynomial ring generated by $\{x_1, \ldots, x_n\}$ and subject to the relations $x_j x_i = q x_i x_j$ for all $i < j$. Suppose q is a primitive ℓth root of unity for some $\ell > 1$. If n is odd, then there is an automorphism which is elementary and (-1)-affine, but not affine:

$$x_i \mapsto \begin{cases} x_i & \text{if } i < n, \\ x_n + x_1^{\ell-1} x_2 \cdots x_{n-2}^{\ell-1} x_{n-1} & \text{if } i = n. \end{cases}$$

On the other hand, if n is even, then every automorphism of $k_q[x_1, \dots, x_n]$ is affine: see the next section.

The Nagata automorphism of the ordinary polynomial algebra $k[x_1, x_2, x_3]$ is (-2)-affine but not a composite of (-1)-affine automorphisms [16].

The definition of a $(-s)$-affine automorphism (and that of a $(-s)$-dominating element) depends on Y (or on the filtration of A). But in most cases, there is an obvious choice of filtration.

We conclude this section by proving Theorem 1.13. This is a generalization of the main result of [6], namely, [6, Theorem 3]. We need to develop a few tools, first. Let R be a central subalgebra of A and let F be a ring of fractions of R (for example, the field of fractions of R). Write $A_F := A \otimes_R F$ and suppose that A_F is finitely generated free over F.

Here is a list of linear algebra facts without proof.

Lemma 1.9. *Suppose that A_F is finitely generated free over F and that v is a positive integer. Let* tr *be the regular trace map* $\mathrm{tr} : A_F \to F$. *Let $Z := \{z_i\}_{i=1}^v$ and $Z' := \{z'_i\}_{i=1}^v$ be subsets of A, and suppose $y_1 \in A$.*

(1) *Let $Z_2 = \{y_1, z_2, \dots, z_v\}$ and $Z_3 = \{y_1 + z_1, z_2, \dots, z_v\}$. Then*

$$d_v(Z_3, Z' : \mathrm{tr}) = d_v(Z, Z' : \mathrm{tr}) + d_v(Z_2, Z' : \mathrm{tr}).$$

(2) *If Z is linearly dependent over F, then $d_v(Z, Z' : \mathrm{tr}) = 0$.*
(3) *If $Z_1 = \{cz_1, z_2, \dots, z_v\}$ for $c \in F$, then $d_v(Z_1, Z' : \mathrm{tr}) = cd_v(Z, Z' : \mathrm{tr})$.*
(4) *Let X be a generating set of A over R. Then $d_v(Z, Z' : \mathrm{tr})$ is an R-linear combination of elements $d_v(X_1, X_2 : \mathrm{tr})$, where X_1 and X_2 consist of v elements in X.*

Definition 1.10. A subset $b = \{b_1, \dots, b_w\} \subset A$ is called a *semi-basis* of A if it is an F-basis of A_F, where b_i is viewed as $b_i \otimes 1 \in A_F$. In this case w is the rank of A over R. The set b is called a *quasi-basis* of A (with respect to X) if

(1) $b = \{b_1, \dots, b_w\}$ is a semi-basis of A, and
(2) There is a set of elements $X = \{x_j\}_{j \in J}$ containing b such that A is generated by X as an R-module and every element $x_j \in X$ is of the form cb_i for some $c \in F$ and $b_i \in b$. We denote the element c by $(x_j : b_i)$.

Let $Z := \{z_1, \dots, z_w\}$ be a subset of A. If b is a semi-basis, then for each i,

$$z_i = \sum_{j=1}^{w} a_{ij} b_j \quad \text{for some } a_{ij} \in F.$$

The $w \times w$-matrix (a_{ij}) is denoted by $(Z : b)$. Let X be a set of generators of A as an R-module, and assume that X contains b. Let X/b denote the subset of F consisting of nonzero scalars of the form $\det(Z : b)$ for all $Z \subset X$ with $|Z| = w$. Let

$$\mathcal{D}(X/b) = \{d_w(b : \mathrm{tr}) f f' \mid f, f' \in X/b\}.$$

Note that if Z and Z' are w-element subsets of X, then

$$\begin{aligned} d_w(Z, Z' : \mathrm{tr}) &= \det(\mathrm{tr}(z_i z'_j)) = \det((Z : b)(\mathrm{tr}(b_i b_j))(Z' : b)^t) \\ &= \det(Z : b) \det(Z' : b) \det(\mathrm{tr}(b_i b_j)) \\ &= \det(Z : b) \det(Z' : b) d_w(b : \mathrm{tr}) \in \mathcal{D}(X/b). \end{aligned} \tag{1.10.1}$$

For any integer v, define

$$\mathcal{D}_v(X) = \{d_v(Z, Z' : \mathrm{tr}) \mid Z, Z' \subset X\}.$$

Then $\mathcal{D}_w(X) = \mathcal{D}(X/b)$. As a consequence of (1.10.1), if Z and Z' are two R-bases of A, then

$$d_w(Z, Z' : \mathrm{tr}) =_{R^\times} d_w(b : \mathrm{tr}). \tag{1.10.2}$$

If $b = \{b_1, \ldots, b_w\}$ is a quasi-basis with respect to $X = \{x_j\}_{j \in J}$, then for each i, let C_i be the set of nonzero elements of the form $(x_j : b_i)$ for all j. It is easy to see that every element in X/b is of the form $c_1 c_2 \cdots c_w$, where $c_i \in C_i$ for each i. Let

$$\mathcal{D}^c(X/b) = \{d_w(b : \mathrm{tr}) \prod_{i=1}^{w} (c_i c'_i) \mid c_i, c'_i \in C_i\}.$$

If b is a quasi-basis with respect to X, then $\mathcal{D}(X/b) = \mathcal{D}^c(X/b)$.

Lemma 1.11. *Let X be a set of generators of A as an R-module and $w = \mathrm{rk}(A/R)$.*

(1) *For any $v \geq 1$, the modified v-discriminant ideal $MD_v(A : \mathrm{tr})$ is generated by $d_v(Z, Z' : \mathrm{tr})$ for all $Z, Z' \subset X$.*
(2) *For any $v \geq 1$, the v-discriminant $d_v(A/R)$ is the gcd of $\mathcal{D}_v(X)$.*
(3) *If b is a semi-basis of A, then $d_w(A/R) = \gcd \mathcal{D}(X/b)$.*
(4) *If b is a quasi-basis of A with respect to X, then $d_w(A/R) = \gcd \mathcal{D}^c(X/b)$.*

Proof. (1) This follows from Lemma 1.9(4).

(2), (3) and (4) follow from the definition and part (1). □

Let C be an algebra. We say that $A \otimes C$ is A-*closed* if, for every $0 \neq f \in A$ and $x, y \in A \otimes C$, the equation $xy = f$ implies that $x, y \in A$ up to units of $A \otimes C$. For example, if C is connected graded and $A \otimes C$ is a domain, then $A \otimes C$ is A-closed.

Lemma 1.12. *Let C be a k-flat commutative algebra such that $A \otimes C$ is a domain and let v be a positive integer.*

(1) $MD_v(A \otimes C : \mathrm{tr} \otimes C) = MD_v(A : \mathrm{tr}) \otimes C$.
(2) *Suppose $A \otimes C$ is A-closed. If $d_v(A/R)$ exists, then $d_v(A \otimes C/R \otimes C)$ exists and equals $d_v(A/R)$.*

Proof. (1) Let X be a set of generators of A as an R-module. Then X is also a set of generators of $A \otimes C$ as an $R \otimes C$-module. The assertion follows from Lemma 1.11(1).

(2) Suppose $d := d_v(A/R)$ exists. Then it is the gcd of $d_v(Z, Z' : \mathrm{tr})$ in A for all $Z, Z' \subset X$ (Lemma 1.11(2)). Let d' be a common divisor of $d_v(Z, Z' : \mathrm{tr})$ in $A \otimes C$ for all $Z, Z' \subset X$. Then we may assume that d' is in A by the A-closedness of $A \otimes C$. Hence d' divides d. Therefore d is the gcd of $\{d_v(Z, Z' : \mathrm{tr}) \mid Z, Z' \subset X\}$ in $A \otimes C$. The assertion follows from Lemma 1.11(2). $\square$

As before let A be a filtered algebra with standard filtration $F_n A = (k \oplus Y)^n$, where $Y := \bigoplus_{i=1}^n kx_i$ generates A, and assume that the associated graded ring $\mathrm{gr}\, A$ is a connected graded domain. Let $C(A)$ denote the center of A. The discriminant of A can also control the automorphism group of $A[t]$. For any $g \in \mathrm{Aut}(A)$, $c \in k^\times$ and $r \in C(A)$, the map

$$\sigma : t \mapsto ct + r, \quad x \mapsto g(x) \quad \text{for all } x \in A \tag{1.12.1}$$

determines uniquely a *triangular* automorphism of $A[t]$. The non-affine automorphisms given in Example 1.8 can be viewed as elementary triangular automorphisms of the Ore extension $D[x_n; \tau]$, where D is the subalgebra generated by $\{x_1, \ldots, x_{n-1}\}$. We associate the triangular automorphism σ (1.12.1) with the upper triangular matrix $\begin{pmatrix} g & r \\ 0 & c \end{pmatrix}$. The triangular automorphisms form a subgroup of $\mathrm{Aut}(A[t])$, denoted by $\begin{pmatrix} \mathrm{Aut}(A) & C(A) \\ 0 & k^\times \end{pmatrix}$ or $\mathrm{Aut}_{\mathrm{tr}}(A[t])$. Explicit examples are computed in [6, Theorems 4.10 and 4.11].

Now we are ready to prove Theorem 1, which is a discriminant criterion for affine automorphisms.

Theorem 1.13. *Let A be an algebra and let Y be a k-subspace of A which generates A as an algebra. Give A the standard filtration $F_n A = (k+Y)^n$ and suppose that the associated graded ring $\mathrm{gr}\, A$ is a connected graded domain. Suppose also that A has finite rank over its center $C(A)$. Assume that there is an integer $v \geq 1$ such that the v-discriminant $d_v(A/C(A))$ is locally dominating with respect to Y. In parts (2–5) we further assume that $d_v(A/C(A))$ is dominating with respect to Y. Then the following hold.*

(1) *Every automorphism of A is affine.*
(2) $\mathrm{Aut}(A[t]) = \mathrm{Aut}_{\mathrm{tr}}(A[t])$.

Suppose that $\mathbb{Z} \subset k$ *in parts* (3,4,5) *and further that* k *is a field in part* (5).

(3) *Every locally nilpotent derivation* ∂ *of* $A[t]$ *is of the form*

$$\partial(x) = 0 \quad \textit{for all } x \in A, \quad \partial(t) = r \quad \textit{for some } r \in C(A).$$

(4) *Every locally nilpotent derivation of* A *is zero.*

(5) $\operatorname{Aut}(A)$ *is an algebraic group that fits into an exact sequence*

$$1 \to (k^{\times})^r \to \operatorname{Aut}(A) \to S \to 1$$

for some finite group S. *Indeed,* $\operatorname{Aut}(A) = S \ltimes (k^{\times})^r$.

Proof. (1) Let $g \in \operatorname{Aut}(A)$. By Lemma 1.4(4), $d_v(A/C(A))$ is g-invariant. By [6, Lemma 2.6], g is affine.

(2) Note that $A \otimes k[t]$ is A-closed (taking $C = k[t]$), and $(A[t])^{\times} = A^{\times}$. By Lemma 1.12(2), $d_v(A/C(A)) =_{A^{\times}} d_v(A[t]/C(A[t]))$. Then the proof of [6, Lemma 3.2] works for $d_v(A/C(A))$. Let $h \in \operatorname{Aut}(A[t])$. By [6, Lemma 3.2(2)], $h(x_i) \in Y \oplus k \subset A$, or $h(A) \subset A$. Applying [6, Lemma 3.2(2)] to $h' := h^{-1}$, we have $h'(A) \subset A$. Thus $h|_A$ and $h'|_A$ are inverse to each other and hence $h|_A \in \operatorname{Aut}(A)$. The rest is the same as the proof of [6, Theorem 3.5].

(3), (4) and (5). By localizing the commutative domain k, we may assume that k is a field of characteristic zero. The rest of the proof follows from the proof of [6, Theorem 3.5(2,3,4)]. □

In this paper we only consider standard filtrations. As explained in [6, Example 5.8], the ideas presented here may be applied to non-standard filtrations.

2. The discriminant and skew polynomial rings

In the first half of this section we discuss some properties related to the center of skew polynomial rings. In the second half of the section, we compute the discriminant of the skew polynomial ring over its center.

Recall that the skew polynomial ring $k_{p_{ij}}[x_1, \ldots, x_n]$ is a connected graded Koszul algebra that is generated by x_i with $\deg x_i = 1$, and subject to the quadratic relations $x_j x_i = p_{ij} x_i x_j$ for all $i < j$, where $p_{ij} \in k^{\times}$ for all $i < j$. We also write $k_{p_{ij}}[\underline{x}_n]$ for the skew polynomial ring $k_{p_{ij}}[x_1, \ldots, x_n]$. It is well known that, if k is a field, then $k_{p_{ij}}[\underline{x}_n]$ is a noetherian domain of Gelfand–Kirillov dimension, Krull dimension, and global dimension n [14]. If the parameters p_{ij} are generic, then $\operatorname{Aut}(k_{p_{ij}}[\underline{x}_n]) = (k^{\times})^n$ [2,22]. In this paper we are interested in the case when the p_{ij} are not generic.

Consider the following two conditions:

(H1) x_i is not central in $k_{p_{ij}}[\underline{x}_n]$ for all $i = 1, \ldots, n$.

(H2) p_{ij} is a root of unity for all $i < j$.

Note that (H2) is equivalent to the following:

(H2') There are positive integers ϕ_{ij} and a primitive root of unity q such that $p_{ij} = q^{\phi_{ij}}$ for all $i < j$; the element q is a generator of the subgroup of $k^\times$ generated by $\{\phi_{ij}\}$.

Throughout the rest of this section let $A = k_{p_{ij}}[\underline{x}_n]$. Note that every monomial $x_1^{d_1} \cdots x_n^{d_n}$ is normal in A. Condition (H1) ensures that A is not a commutative polynomial ring. Condition (H2) implies that A is PI. Since A is $\mathbb{Z}^n$-graded with $\deg x_i = (0, \ldots, 1, \ldots, 0)$, where 1 is in the ith position, the center of A is $\mathbb{Z}^n$-graded. Thus the center of A has a k-linear basis consisting of monomials.

Definition 2.1. For each i, define an automorphism ϕ_i of A, called a *conjugation automorphism* or *conjugation by* x_i, by

$$\phi_i(x_j) = p_{ij} x_j \quad \forall\, i, j$$

(where, as earlier, $p_{ii} = 1$ for all i and $p_{ij} = p_{ji}^{-1}$ if $i > j$).

For each monomial $f := x_1^{d_1} \cdots x_n^{d_n}$, $\phi_i(f) = \prod_{j=1}^n p_{ij}^{d_j} f$. Hence $\phi_i(f) = f$ if and only if $\prod_{j=1}^n p_{ij}^{d_j} = 1$. Since ϕ_i is conjugation by x_i, x_i commutes with f if and only if $\phi_i(f) = f$, and then if and only if $\prod_{j=1}^n p_{ij}^{d_j} = 1$. Define

$$T = \{(d_1, \ldots, d_n) \in \mathbb{N}^n \mid \prod_{j=1}^n p_{ij}^{d_j} = 1\ \forall\, i\}.$$

If W is any subset of $\mathbb{N}^n$, let

$$X^W = \{x_1^{d_1} \cdots x_n^{d_n} \mid (d_1, \ldots, d_n) \in W\}.$$

Lemma 2.2. *Retain the above notation. Then the following hold.*

(1) *The center $C(A)$ of A has a monomial basis $\{f \mid f \in X^T\}$.*
(2) *Assume* (H2) *and that k is a field. Then $C(A)$ is Cohen–Macaulay.*

Proof. (1) This is clear.

(2) If $\operatorname{char} k = 0$, this is well known [17, Theorem 2.2(3)]. Now we assume that $\operatorname{char} k = p > 0$. Let S be the abelian group generated by the conjugation automorphisms ϕ_i. Then S is a finite group and $C(A)$ is the fixed subring A^S. The order of ϕ_i equals the order of the subgroup G of $k^\times$ generated by $\{p_{i1}, p_{i2}, \ldots, p_{in}\}$. Since G is a subgroup of $k^\times$, it is cyclic. We may assume that the base field k is finite. Then $|k| = p^N$ for some N and $k^\times$ is a cyclic group of order $p^N - 1$. Thus the order of G is coprime to p. Since each ϕ_i has order coprime to p, the order of S is coprime to p. As a consequence, the group

algebra kS is semisimple. Then A^S is Cohen–Macaulay by [12, Lemma 3.2(b)] (note that the proof of [12, Lemma 3.2(b)] only uses the fact kS is semisimple, not the hypothesis $\operatorname{char} k = 0$). □

When n is large, it is not easy to understand $C(A)$ or T completely. The following lemma is useful in a special case.

Lemma 2.3. *Assume* (H2). *The following are equivalent.*

(1) *The center $C(A)$ is a polynomial ring.*
(2) *There are positive integers $a_1, \dots, a_n$ such that $(d_1, \dots, d_n) \in T$ if and only if $a_i \mid d_i$ for all i. In other words, T is generated by $(0, \dots, 0, a_i, 0, \dots, 0)$ for $i = 1, \dots, n$, where a_i is in the ith position, and the $\mathbb{N}^n$-solutions $(d_1, \dots, d_n)$ to the system of equations*

$$\prod_{j=1}^{n} p_{ij}^{d_j} = 1, \quad \text{for all } i = 1, \dots, n \tag{2.3.1}$$

form the set $\{\sum_{i=1}^{n} b_i(0, \dots, 0, a_i, 0, \dots, 0) \mid b_i \geq 0\}$.
(3) *There are positive integers $a_1, \dots, a_n$ such that $C(A)$ is generated by $x_i^{a_i}$ for $i = 1, \dots, n$.*
(4) *A is finitely generated free over $C(A)$.*

Proof. (1) ⇒ (2) By localizing k, we may assume that k is a field. Since $C(A)$ and A have the same Gelfand–Kirillov dimension, the number of generators in $C(A)$ must be n. Let a_i be the minimal integer such that $(0, \dots, 0, a_i, 0, \dots, 0) \in T$. Then $x_i^{a_i} \in C(A)$, but is not generated by any other elements in $C(A)$. Let $\mathfrak{m}$ be the graded ideal $C(A)_{\geq 1}$. Then the images of $x_i^{a_i}$ in $\mathfrak{m}/\mathfrak{m}^2$ (still denoted by $x_i^{a_i}$) are linearly independent elements; since $\mathfrak{m}/\mathfrak{m}^2$ is a free k-module of rank n, we have $\mathfrak{m}/\mathfrak{m}^2 = \oplus_{i=1}^{n} k x_i^{a_i}$. Thus $C(A)$ is generated by $x_i^{a_i}$. The assertion follows.

(2) ⇒ (3) and (3) ⇒ (4) are clear.

(4) ⇒ (1) Let F be the field of fractions of k. Then $A \otimes F$ is finitely generated free over $C(A) \otimes F$. Since F is a field, $C(A) \otimes F$ has global dimension n [11, Lemma 1.11]. The only connected graded commutative algebra of finite global dimension is the polynomial ring. So $C(A) \otimes F$ is a polynomial ring. By the proof of (1) ⇒ (2) for $k = F$, $C(A) \otimes F$ is generated by $x_i^{a_i}$s. Therefore $C(A)$ is generated by $x_i^{a_i}$. Thus $C(A)$ is a polynomial ring. □

Example 2.4. Let q be a primitive ℓth root of unity, and write $p_{ij} = q^{\phi_{ij}}$ for some integers ϕ_{ij}.

(1) If $\det(\phi_{ij})$ is invertible in $\mathbb{Z}/(\ell)$, then the center of A is $k[x_1^{\ell}, \dots, x_n^{\ell}]$. To see this, let $x_1^{d_1} \cdots x_n^{d_n}$ be in the center. By the definition of T, we have $\prod_{j=1}^{n} p_{ij}^{d_j} = 1$ for all i, or equivalently,

$$\sum_{j=1}^{n} \phi_{ij} d_j \equiv 0 \pmod{\ell}.$$

Since $\det(\phi_{ij})_{n\times n}$ is invertible in $\mathbb{Z}/(\ell)$, $d_i \equiv 0 \pmod{\ell}$, or $\ell \mid d_i$. It is clear that $x_i^\ell \in C(A)$. Thus $C(A) = k[x_1^\ell, \dots, x_n^\ell]$.

Note that we may take $\phi_{ii} = 0$ and $\phi_{ji} = -\phi_{ij}$. Then the matrix (ϕ_{ij}) is skew-symmetric. Hence $\det(\phi_{ij})$ being invertible can only happen when n is even.

(2) A special case of (1) is when $\phi_{ij} = 1$ for all $i < j$ (or $p_{ij} = q$ for all $i < j$). When n is even, then, by linear algebra, $\det(\phi_{ij}) = 1$, which is invertible for any ℓ. In this case the center of $k_q[x_1, \dots, x_n]$ is $k[x_1^\ell, \dots, x_n^\ell]$.

(3) When n is odd, there are different kinds of examples for which $C(A)$ is a polynomial ring. Let $n = 3$ and q be a primitive ℓth root of unity. Suppose $\ell = abc$, where $a, b, c \geq 2$ are pairwise coprime. Let $p_{12} = q^{ab}$, $p_{13} = q^{ac}$, and $p_{23} = q^{bc}$. Then one can check that the center of $k_{p_{ij}}[x_1, x_2, x_3]$ is $k[x_1^{bc}, x_2^{ac}, x_3^{ab}]$. Higher dimensional examples can be constructed in a similar way.

(4) Again let $n = 3$, q be a primitive ℓth root of unity, and $\ell = abc$, where $a, b, c \geq 2$ are pairwise coprime. Let $p_{12} = q^a$, $p_{13} = q^{-b}$, and $p_{23} = q^c$. Then the center $C(A)$ is not a polynomial ring. To see this, note that the monomials $x_1^\ell, x_2^\ell, x_3^\ell, x_1^c x_2^b x_3^a$, and so on, are generators of $C(A)$, but x_1^c is not in the center. By Lemma 2.3, $C(A)$ is not a polynomial ring, and in this case, $C(A) \subset k\langle x_1^c, x_2^b, x_3^a \rangle$.

Note that under the hypothesis (H2), the subgroup of $k^\times$ generated by $\{p_{ij}\}$ is $\langle q \rangle$ for some root of unity q.

Lemma 2.5. *Assume* (H1) *and* (H2), *hence* (H2'). *Assume that the group generated by* $\{p_{ij}\}$ *is* $\langle q \rangle$, *where* q *is a primitive* ℓ*th root of unity and* ℓ *is a prime number. If* $C(A)$ *is not a polynomial ring, then there is a solution* $(d_1, d_2, \dots, d_n) \in \mathbb{N}^n$ *to the system of equations*

$$\prod_{j=1}^{n} p_{ij}^{d_j} = 1, \quad \textit{for all } i = 1, \dots, n$$

such that $d_s = 1$ *for some* s.

Proof. Since $x_i \notin C(A)$ and $x_i^\ell \in C(A)$, we have

$$\ell = \min_{a>0} \{x_i^a \in C(A)\}.$$

Since $C(A)$ is not a polynomial ring, there is a solution $d := (d_1, d_2, \dots, d_n)$ to system of equations given in the lemma such that some d_s is not divisible by ℓ. Note that any multiple of d is still a solution. By replacing d by a multiple of d, we have $d_s \equiv 1 \pmod{\ell}$ (as ℓ is prime). Finally, by replacing d_s by 1 (as $p_{is}^\ell = 1$) we obtain the desired solution. □

Next we compute the discriminant $d(A/R)$ when R is a polynomial ring. We start with an easy lemma. Let Λ be an abelian group and let B be a Λ-graded algebra. Then the center of B is also Λ-graded.

Lemma 2.6. *Let B be a Λ-graded algebra and R a central graded subalgebra of B. Suppose that $R^\times = k^\times$. For every $v \geq 1$ and any sets of homogeneous elements $Z = \{z_i\}_{i=1}^v$ and $Z' = \{z'_i\}_{i=1}^v$, the discriminant $d_v(Z, Z' : \mathrm{tr})$ is either 0 or homogeneous of degree $\sum_{i=1}^v (\deg z_i + \deg z'_i)$. As a consequence, if B is a finitely generated graded free module over R, then $d(B/R)$ is homogeneous.*

Proof. The consequence is clear, so we prove the main assertion.

Let F be the graded field of fractions of $C(B)$. Since $C(B)$ is graded, we can choose a semi-basis $b = \{b_1, \ldots, b_w\}$ of B consisting of homogeneous elements b_i, where $w = \mathrm{rk}(B/C(B))$. Then B is a finitely generated graded free module over F with basis b. For each homogeneous element f, $\mathrm{tr}(f)$ is either 0 or homogeneous of degree $\deg(f)$. In particular, $\mathrm{tr}(z_i z'_j)$ is either 0 or homogeneous of degree $\deg(z_i z'_j) = \deg(z_i) + \deg(z'_j)$. By definition, $d_v(Z, Z' : \mathrm{tr})$ is the determinant $\det(\mathrm{tr}(z_i z'_j))_{v\times v}$, which is a signed sum of elements

$$\sum_{\sigma\in S_v} \mathrm{tr}(z_1 z'_{\sigma(1)})\, \mathrm{tr}(z_2 z'_{\sigma(2)}) \cdots \mathrm{tr}(z_v z'_{\sigma(v)}).$$

Each above element is either 0 or homogeneous of degree $\sum_{i=1}^v (\deg z_i + \deg z'_i)$. Hence $d_v(Z, Z' : \mathrm{tr})$ is either 0 or homogeneous of degree $\sum_{i=1}^v (\deg z_i + \deg z'_i)$. $\square$

We may consider $k_{p_{ij}}[\underline{x}_n]$ as either $\mathbb{Z}$-graded or $\mathbb{Z}^n$-graded. Let $k_{p_{ij}}[\underline{x}_n^{\pm 1}]$ denote the algebra $k_{p_{ij}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. By a monomial in $k_{p_{ij}}[\underline{x}_n^{\pm 1}]$, we mean an element of the form $c x_1^{a_1} \cdots x_n^{a_n}$ for some $a_i \in \mathbb{Z}$ and some $0 \neq c \in k$.

Lemma 2.7. *Let $A = k_{p_{ij}}[\underline{x}_n]$ and $B = k_{p_{ij}}[\underline{x}_n^{\pm 1}]$ with the natural $\mathbb{Z}^n$-grading. Let $C(A)$ be the center of A. In parts* (6)–(9) *suppose* (H2) *and let* $\mathrm{tr} : A \to F$ *be the regular trace, where F is the field of fractions of $C(A)$.*

(1) *Every homogeneous element in B is a monomial.*
(2) *Let f be a homogeneous element in B. Then $f \in A$ if and only if $\deg f \in \mathbb{N}^n$.*
(3) *For any set $\mathcal{D}$ of monomials in A, $\gcd \mathcal{D}$ exists and is a monomial.*
(4) *The center of A (respectively, B) is a $\mathbb{Z}^n$-graded subalgebra of A (respectively, B).*
(5) *There is a generating set X of the $C(A)$-module A consisting of monomials.*
(6) *If A satisfies* (H2)*, then A has a quasi-basis b.*
(7) *The rank $\mathrm{rk}(A/C(A))$ is nonzero in k.*
(8) *For every monomial $f \in B$, $\mathrm{tr}(f) \neq 0$ if and only if $f \in C(B)$.*
(9) *The image of $\mathrm{tr} : A \to F$ is in $C(A)$.*

Proof. (1)–(5) are straightforward.

(6) Since B is a graded division ring, its center is a graded field. Hence B is finitely generated graded free over $C(B)$ with a basis $b \subset X$. It is easy to check that b is a quasi-basis.

(7) Since $\mathrm{rk}(A/C(A)) = \mathrm{rk}(B/C(B))$, it suffices to show that $\mathrm{rk}(B/C(B))$ is nonzero in k. By localizing k, we may assume that k is a field. If $\mathrm{char}\, k = 0$, the assertion is trivial, so we assume that $\mathrm{char}\, k = p > 0$.

Following the proof of Lemma 2.2(2), let S be the abelian group generated by the automorphisms ϕ_i. Then S is a finite group and $C(B) = B^S$. Since each p_{ij} is a root of unity, by replacing k by the subfield generated by the p_{ij}'s, we may assume k is finite. Then $|k| = p^N$ for some N. By the proof of Lemma 2.2(2), the order of S is coprime to p. Since $C(B)$ is a $\mathbb{Z}^n$-graded field, B is a finite dimensional free module over $C(B)$ with a monomial basis $b = \{b_1 = 1, \ldots, b_w\}$. Let $S^\vee$ be the dual group of S. Define a map $\Phi : b \to S^\vee$ by $\Phi(b_i)(\phi_j) = \phi_j(b_i)b_i^{-1}$. If $b_i b_j = b_k c$ for some $c \in C(B)$, then one can check that $\Phi(b_i)\Phi(b_j) = \Phi(b_k)$. This observation implies that Φ is injective and the image of Φ is a subgroup of $S^\vee$. Therefore the order of b, namely, $\mathrm{rk}(B/C(B))$, is a divisor of $|S|$, which is coprime to p. Equivalently, $\mathrm{rk}(B/C(B)) \neq 0$ in k.

(8) The regular trace map $\mathrm{tr} : A \to F$ (or $\mathrm{tr} : B \to F$) can be defined by composing

$$\mathrm{tr} : A \to A \otimes_{C(A)} C(B) = B \xrightarrow{lm} M_w(C(B)) \xrightarrow{\mathrm{tr_{int}}} C(B) \xrightarrow{=} F,$$

where lm is the left multiplication map. For any monomial f in A (or in B), $tr(f)$ is either zero or of degree equal to $\deg(f)$ – that is, the map tr is homogeneous of degree 0 with respect to the $\mathbb{Z}^n$-grading. Thus if $\mathrm{tr}(f) \neq 0$, then $\mathrm{tr}(f) \in C(B)$ is a scalar multiple of $f \in B$, so f is in $C(B)$. If $f \in C(B)$, then $\mathrm{tr}(f) = wf$, where $w = \mathrm{rk}(A/C(A))$ is nonzero in k, by part (7).

(9) Since the map tr is homogeneous of degree 0, the image $\mathrm{im}\,\mathrm{tr}(A)$ is in A by part (2). Hence $\mathrm{im}\,\mathrm{tr}(A) \subset A \cap C(B) = C(A)$. □

Proposition 2.8. *Consider A as a $\mathbb{Z}^n$-graded algebra. Let $R = k[x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}]$ be a central subalgebra of A, where the α_i are positive integers. Let $r = \prod_{i=1}^n \alpha_i$. Then*

$$d(A/R) =_{k^\times} r^r (\prod_{i=1}^n x_i^{\alpha_i - 1})^r.$$

As a consequence, if R is the center of A and $\alpha_i > 1$ for all i, then $d(A/R)$ is dominating.

Proof. First note that there is a graded basis $Z := \{x_1^{\beta_1} \cdots x_n^{\beta_n} \mid 0 \le \beta_i < \alpha_i \ \forall i\}$ of A over R, so the rank of A over R is $r = \prod_{i=1}^n \alpha_i$.

Let $b := \{z_1 = 1, z_2, \ldots, z_r\}$ be a monomial basis of A over R. For every element $z_j := x_1^{\beta_1} \cdots x_n^{\beta_n}$ in the basis b, let z_j' be the monomial $x_1^{\beta_1'} \cdots x_n^{\beta_n'}$, where

$$\beta_i' = \begin{cases} 0 & \text{if } \beta_i = 0, \\ \alpha_i - \beta_i & \text{if } \beta_i \neq 0. \end{cases}$$

One can check that z_j' is the unique element in the basis such that $z_j z_j' \in R$. For example, $z_1' = 1$. Then $\operatorname{tr}(z_j z_s) = 0$ unless $z_s = z_j'$, and in that case $\operatorname{tr}(z_j z_j') = r z_j z_j'$. Therefore $\det(\operatorname{tr}(z_i z_j)) =_{k^\times} r^r \prod_{j=1}^r z_j z_j'$. An easy combinatorial argument gives the result.

For the consequence, note that the rank r is nonzero in k by Lemma 2.7(7). Then $d(A/R)$ is of the form given in [6, Lemma 2.2(1)], which is dominating. □

From Lemma 2.3, we see that if the center of A is a commutative polynomial ring, then the center is of the form $k[x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}]$. So as an immediate consequence of this, together with Proposition 2.8 and Theorem 1, if (H2) holds and if the center of A is a commutative polynomial ring, then every automorphism of A is affine and every automorphism of $A[t]$ is triangular.

We also consider the discriminant when $C(A)$ is not a polynomial ring. The goal is an explicit condition that ensures that the discriminant is dominating. We recall some notation. Fix a parameter set $\{p_{ij} \mid 1 \leq i < j \leq n\}$ and impose the usual conditions ($p_{ji} = p_{ij}^{-1}$, $p_{ii} = 1$) to define p_{ij} for all $1 \leq i, j \leq n$. For any $1 \leq s \leq n$, let

$$T_s = \{(d_1, \ldots, \widehat{d_s}, \ldots, d_n) \in \mathbb{N}^{n-1} \mid \prod_{\substack{j=1 \\ j \neq s}}^{n} p_{ij}^{d_j} = p_{is} \ \forall\, i \neq s\}.$$

Lemma 2.9. *Retain the above notation.*

(1) *If $(d_1, \ldots, \widehat{d_s}, \ldots, d_n) \in T_s$, then the equation $\prod_{j=1, j\neq s}^{n} p_{ij}^{d_j} = p_{is}$ also holds for $i = s$.*
(2) $T_s = \{(d_1, \ldots, \widehat{d_s}, \ldots, d_n) \in \mathbb{N}^{n-1} \mid x_1^{d_1} \cdots x_s^{-1} \cdots x_n^{d_n} \in C(k_{p_{ij}}[\underline{x}_n^{\pm 1}])\}$.

Proof. Both are easy to check. □

By Lemma 2.9(1), $(d_1, \ldots, \widehat{d_s}, \ldots, d_n) \in T_s$ if and only if $(d_1, \ldots, \widehat{d_s}, \ldots, d_n)$ is an $\mathbb{N}^{n-1}$-solution to the system of equations

$$\prod_{\substack{j=1 \\ j \neq s}}^{n} p_{ij}^{d_j} = p_{is}, \ \forall\, i. \tag{2.9.1}$$

The next lemma is easy and the proof is omitted.

Lemma 2.10. *Let B be a Λ-graded domain, where Λ is a linearly ordered group. Let c be a homogeneous element in B and $a, b \in B$ such that $ab = c$. Then both a and b are homogeneous.*

For $d := (d_1, \ldots, \widehat{d_s}, \ldots, d_n)$ in T_s, define $f_d = x_1^{d_1} \cdots \widehat{x}_s \cdots x_n^{d_n}$. Then by Lemma 2.9(2), $x_i f_d = p_{si} f_d x_i$ for all i. Therefore the map

$$g(cf_d, s) : x_i \mapsto \begin{cases} x_i & \text{if } i \neq s, \\ x_s + cf_d & \text{if } i = s \end{cases} \tag{2.10.1}$$

extends to an algebra automorphism of A, where $c \in k$. The map

$$\partial(cf_d, s) : x_i \mapsto \begin{cases} 0 & \text{if } i \neq s, \\ cf_d & \text{if } i = s \end{cases} \tag{2.10.2}$$

extends to a locally nilpotent derivation of A. By slight abuse of notation, we let

$$X^{T_s} = \{x_1^{d_1} \cdots \widehat{x}_s \cdots x_n^{d_n} \mid (d_1, \ldots, \widehat{d_s}, \ldots, d_n) \in T_s\}.$$

If F is a linear combination of monomials in X^{T_s}, we can define $g(F, s)$ and $\partial(F, s)$ similarly. Automorphisms of the form $g(F, s)$ are called *elementary* automorphisms. It is easy to check that $g(F, s)g(F', s) = g(F + F', s)$ as long as both F and F' are linear combinations of monomials in X^{T_s}. As a consequence, $g(F, s)^{-1} = g(-F, s)$.

Theorem 2.11. *Let $A = k_{p_{ij}}[\underline{x}_n]$ be a skew polynomial ring satisfying* (H2). *Let $w = \operatorname{rk}(A/C(A))$.*

(1) *For any positive integer v, the v-discriminant $d_v(A/C(A))$ exists. Furthermore, $d_w(A/C(A))$ is nonzero.*
(2) *For any $1 \leq s \leq n$, $T_s = \emptyset$ if and only if $x_s \mid d_w(A/C(A))$.*
(3) *$T_i = \emptyset$ for all $i = 1, \ldots, n$ if and only if $d_w(A/(C(A))$ is dominating.*

Proof. (1) By Lemma 2.7(5), there is a generating set X of A over $C(A)$ consisting of monomials. For any v-element subsets $Z, Z' \subset X$, $d_v(Z, Z' : \operatorname{tr})$ is homogeneous by Lemma 2.6, and is a monomial in A by Lemma 2.7(2). Applying Lemma 2.7(3) to the set of monomials of the form $d_v(Z, Z' : \operatorname{tr})$ for all such Z and Z', we see that $d_v(A/C(A))$ exists.

For the second assertion it suffices to show that there are Z, Z' such that

$$d_w(Z, Z' : \operatorname{tr}) \neq 0,$$

as $d_w(A/C(A))$ is the gcd of such elements. Let $Z = \{z_1 = 1, z_2, \ldots, z_w\}$ be a quasi-basis of A. For each i, define $z'_i \in A$ to be a nonzero monomial such that $z_i z'_i \in C(A)$, and let $Z' = \{z'_i\}_{i=1}^{w}$. Then $z_i z'_j \notin C(A)$ for all $i \neq j$, whence by Lemma 2.7(8),

$$\operatorname{tr}(z_i z'_j) = \begin{cases} w z_i z'_i & i = j, \\ 0 & i \neq j. \end{cases}$$

Hence $d_w(Z, Z' : \mathrm{tr}) = w^w \prod_{i=1}^{w}(z_i z_i')$ which is a nonzero monomial as $w \neq 0$ (Lemma 2.7(7)).

(2) By using Lemma 2.9(2), if T_s is empty, then $x_1^{d_1} \cdots x_s^{-1} \cdots x_n^{d_n}$ is not in $C(B)$ for any $d_i \in \mathbb{N}$ for all $i \neq s$ (with B as defined in Lemma 2.7).

Let $b = \{b_1, \ldots, b_w\}$ be a quasi-basis with respect to a generating set X (Lemma 2.7(6)); we may assume that X contains x_s. Let $Z = \{z_1, \ldots, z_w\}$ be a subset of X. We claim that x_s divides $d_w(Z, Z' : \mathrm{tr})$ for all Z'. If $d_w(Z, Z' : \mathrm{tr}) = 0$, then the claim follows. If $\det(Z : b) = 0$, then $d_w(Z, Z' : \mathrm{tr}) = 0$, so we assume that $\det(Z : b) \neq 0$. Since $\deg d_w(Z, Z' : \mathrm{tr}) = \deg(\prod_{i=1}^{w} z_i z_i')$ (Lemma 2.6), it's enough to show that x_s divides z_i for some i. Since b is a quasi-basis, up to a permutation, for each i, $z_i = b_i c_i$ for some $0 \neq c_i \in C(B)$. Hence Z is a quasi-basis of A. Therefore, there is an i such that $x_s = z_i c$ for some $c \in C(B)$, or $z_i = x_s c^{-1}$. Since the x_s-degree of c can not be 1, the x_s-degree of $x_s c^{-1}$ is not zero. This means that x_s-degree of z_i is nonzero, or $x_s \mid z_i$.

If T_s is non-empty, pick an element in T_s of the form

$$d' = (d_1' + m\ell, d_2', \ldots, \widehat{d_s'}, \ldots, d_n')$$

with $m \gg 0$. Hence there is a monomial $f_{d'}$ in X^{T_s} with degree larger than the degree of $d := d_w(A/C(A))$. Let $g = g(f_{d'}, s)$ be the automorphism constructed in (2.10.1). Then $\deg g(x_s) > \deg d$. It follows from Lemmas 2.6 and 2.7 that d is homogeneous, whence it is a nonzero monomial, say $c x_1^{a_1} \cdots x_n^{a_n}$. Then we have

$$\deg d = \deg g(d) = \deg\left((g(x_1))^{a_1} \cdots (g(x_n))^{a_n}\right) = \sum_i a_i \deg g(x_i).$$

If $a_s > 0$, then

$$\deg g(x_s) \leq a_s \deg g(x_s) \leq \sum_i a_i \deg g(x_i) = \deg d,$$

which contradicts the fact $\deg g(x_s) > \deg d$. Therefore $a_s = 0$ and x_s does not divide d.

(3) Since $d_w(A/C(A))$ is a monomial, it is of the form $x_1^{a_1} \cdots x_n^{a_n}$, up to a scalar. The assertion follows from part (2). □

Corollary 2.12. *Let $A = k_q[\underline{x}_n]$ be a q-skew polynomial ring and q a primitive ℓth root of unity for some $\ell \geq 2$. Let w be the rank of A over its center. Then*

$$d_w(A/C(A)) = \begin{cases} c \prod_{i=1}^{n} x_i^{\ell^n(\ell-1)} & \text{if } n \text{ is even} \\ c & \text{if } n \text{ is odd,} \end{cases}$$

for some $0 \neq c \in k$. As a consequence, $\mathrm{Aut}(A)$ *is affine if and only if n is even.*

Proof. First we assume that n is even. By Example 2.4(2), the center of A is $k[x_1^\ell, \dots, x_n^\ell]$. Then the discriminant is given by Proposition 2.8. By Theorem 1.13, $\mathrm{Aut}(A)$ is affine. An easy computation gives the formula (0.3.1).

If n is odd, then $(\ell-1, 1, \ell-1, \dots, \widehat{d_s}, \dots, 1, \ell-1) \in T_s$ when s is odd and $(1, \ell-1, \dots, \widehat{d_s}, \dots, \ell-1, 1) \in T_s$ when s is even. By Theorem 2.11(2), $d_w(A/C(A))$ is a constant. By construction (2.10.1), $\mathrm{Aut}(A)$ is not affine. □

3. Affine and tame automorphisms of skew polynomial rings

In this section we reprove and extend some results of Alev and Chamarie about the automorphism groups of skew polynomial rings [2]. Here is one of the main results in this section. Let $\mathrm{LNDer}(B)$ denote the set of all locally nilpotent derivations of an algebra B. As in the previous section, let A be $k_{p_{ij}}[\underline{x}_n]$.

Theorem 3.1. *Let $A = k_{p_{ij}}[\underline{x}_n]$ be a skew polynomial ring satisfying* (H2). *The following are equivalent.*

(1) $\mathrm{Aut}(A)$ *is affine.*
(2) $C(k_{p_{ij}}[\underline{x}_n^{\pm 1}]) \subset k\langle x_1^{\pm\alpha_1}, \dots, x_n^{\pm\alpha_n}\rangle$ *for some* $\alpha_1, \dots, \alpha_n \geq 2$.
(3) $C(k_{p_{ij}}[\underline{x}_n]) \subset k\langle x_1^{\alpha_1}, \dots, x_n^{\alpha_n}\rangle$ *for some* $\alpha_1, \dots, \alpha_n \geq 2$.
(4) $T_s = \emptyset$ *for all* $s = 1, \dots, n$.
(5) $d_w(A/C(A))$ *is dominating where* $w = \mathrm{rk}(A/C(A))$.
(6) $d_w(A/C(A))$ *is locally dominating where* $w = \mathrm{rk}(A/C(A))$.

If $\mathbb{Z} \subset k$, *then the above are also equivalent to*

(7) $\mathrm{LNDer}(A) = \{0\}$.

The proof of Theorem 3.1 is given in the middle of the section. One immediate question is, for what kinds of noetherian connected graded Koszul PI algebras is some version of Theorem 3.1 still valid?

Let B be a connected $\mathbb{N}$-graded algebra generated in degree 1. Let $\mathrm{Aut}_{\mathrm{gr}}(B)$ be the subgroup of graded automorphisms of B. An automorphism g of B is called *unipotent* if $g(v) = v +$ (higher degree terms) for all $v \in B_1$. Let $\mathrm{Aut}_{\mathrm{uni}}(B)$ denote the subgroup of $\mathrm{Aut}(B)$ consisting of unipotent automorphisms.

In what follows, we do not assume (H2) unless explicitly stated.

Lemma 3.2. *The following are equivalent for A.*

(1) *A satisfies* (H1), *namely, x_i is not central for all i.*
(2) *For each i, there is a j such that $p_{ij} \neq 1$.*

(3) *For every commutative domain $C \supseteq k$ and for every k-algebra automorphism g of $A \otimes C$, the constant term of $g(x_i)$ is zero.*
(4) $\operatorname{Aut}(A) = \operatorname{Aut}_{\mathrm{gr}}(A) \ltimes \operatorname{Aut}_{\mathrm{uni}}(A)$.
(5) *For every commutative domain $C \supseteq k$ and for every k-algebra derivation ∂ of $A \otimes C$, the constant term of $\partial(x_i)$ is zero.*
(6) *For every commutative domain $C \supseteq k$ and for every k-algebra locally nilpotent derivation ∂ of $A \otimes C$, the constant term of $\partial(x_i)$ is zero.*

Proof. It is clear that (1) $\Leftrightarrow$ (2) and that (5) $\Rightarrow$ (6).

(3) $\Rightarrow$ (1) If x_i is central, then $g : x_j \to x_j + \delta_{ij}$ defines an algebra automorphism for which the constant term of $g(x_i)$ is not zero.

(2) $\Rightarrow$ (3) Suppose $g \in \operatorname{Aut}(A \otimes C)$ such that $g(x_i) = c_i + y_i$, where $c_i \in C$ is the constant term of $g(x_i)$. Suppose $c_i \neq 0$ for some i. Pick j such that $p_{ij} \neq 1$. Applying g to the equation $x_j x_i = p_{ij} x_i x_j$ we have

$$(c_j + y_j)(c_i + y_i) = p_{ij}(c_i + y_i)(c_j + y_j).$$

By comparing constant terms, we have $c_j c_i = p_{ij} c_i c_j$. Since $p_{ij} \neq 1$ and $c_i \neq 0$, we have $c_j = 0$ (as C is a domain), and

$$y_j(c_i + y_i) = p_{ij}(c_i + y_i)y_j.$$

Let $(y_j)_t$ be the nonzero homogeneous component of the lowest degree part of y_j. Then, by comparing the lowest degree components of the above equation, we have $c_i(y_j)_t = p_{ij} c_i (y_j)_t$. Thus, $(y_j)_t = 0$ as $A \otimes C$ is a domain, contradiction.

(3) $\Leftrightarrow$ (4) Let g be an automorphism of A. Since $g(x_i)$ has zero constant term, $\operatorname{gr} g \in \operatorname{Aut}_{\mathrm{gr}}(A)$ and $g(\operatorname{gr} g)^{-1} \in \operatorname{Aut}_{\mathrm{uni}}(A)$. Hence (4) is equivalent to (3) when $C = k$. Then we use the fact that (3) $\Leftrightarrow$ (1), which is independent of C.

(6) $\Rightarrow$ (1) If x_s is central for some s, then $\partial : x_i \to \delta_{is}, c \to 0$ for all $c \in C$ defines a locally nilpotent derivation such that the constant term of $\partial(x_s)$ is not zero.

(2) $\Rightarrow$ (5) Suppose $\partial(x_i) = c_i + f_i$, where $c_i \in C$ is the constant term of $\partial(x_i)$. Suppose $c_s \neq 0$. Applying ∂ to the equation $x_i x_s = p_{si} x_s x_i$ for $i \neq s$ we have

$$(c_i + f_i)x_s + x_i(c_s + f_s) = p_{si}((c_s + f_s)x_i + x_s(c_i + f_i)).$$

The degree 1 part of the above equation is

$$c_i x_s + c_s x_i = p_{is}(c_s x_i + c_i x_s).$$

Since $p_{is} \neq 1$ for some i, we have $c_s x_i + c_i x_s = 0$, which contradicts $c_s \neq 0$. Therefore the assertion holds. $\square$

By Lemma 3.2(4), to describe $\operatorname{Aut}(A)$, we need understand both $\operatorname{Aut}_{\mathrm{gr}}(A)$ and $\operatorname{Aut}_{\mathrm{uni}}(A)$. The next theorem takes care of $\operatorname{Aut}_{\mathrm{uni}}(A)$ for many cases; this can be viewed as an extension of results in [2], as we give some necessary and sufficient conditions so that $\mathrm{Aut}(A) = \mathrm{Aut}_{\mathrm{gr}}(A)$.

Let $(T_s)_{\geq 2}$ be the subset of T_s consisting of elements $(d_1, \dots, \widehat{d_s}, \dots, d_n)$ with $\sum_j d_j \geq 2$. Recall that $X^W = \{x_1^{d_1} \cdots x_n^{d_n} \mid (d_1, \dots, d_n) \in W\}$.

Let C be a commutative domain containing k and let $g \in \mathrm{Aut}_{\mathrm{uni}}(A \otimes C)$. For each fixed s, write

$$g(x_s) = x_s(1 + h') + g_s,$$

where g_s is in the subalgebra generated by C and $x_1, \dots, \widehat{x}_s, \dots, x_n$. If $g_s \neq 0$, it is further decomposed as

$$g_s = h_{t_s} + \text{higher degree terms},$$

where t_s is the lowest possible degree of a nonzero homogeneous component of g_s. Define a bigrading on g by $\deg g = (a, b)$, where

$$a = \min\{t_s \mid g_s \neq 0 \text{ and } 1 \leq s \leq n\}$$

and $b = \min\{s \mid t_s = a\}$. If $g_s = 0$ for all s, then we write $\deg g = (-\infty, -\infty)$. Otherwise, $\deg g \in \{2, 3, 4, \dots\} \times \{1, 2, \dots, n\}$. For pairs of integers (a_1, b_1) and (a_2, b_2), we define $(a_1, b_1) < (a_2, b_2)$ if either $a_1 < a_2$ or $a_1 = a_2$ and $b_1 < b_2$.

Lemma 3.3. *Let $g, g_1, g_2 \in \mathrm{Aut}_{\mathrm{uni}}(A)$.*

(1) *If $\deg g = (-\infty, -\infty)$, then g is the identity.*
(2) *$\deg g_1 g_2 \geq \min\{\deg g_1, \deg g_2\}$ and equality holds if $\deg g_1 \neq \deg g_2$.*
(3) *If g is not the identity, there is $(d_1, \dots, \widehat{d_s}, \dots, d_n) \in (T_s)_{\geq 2}$ and F, a linear combination of $f_d \in X^{(T_s)_{\geq 2}}$ of the same total degree, such that $\deg g(F, s) = \deg g$ and $\deg g(F, s)g > \deg g$. (Here, $g(F, s)$ is as defined in (2.10.1).)*

Proof. (1) Since $\deg g = (-\infty, -\infty)$, by definition, $g_s = 0$ for all s, or $g(x_s) = x_s(1+h')$, where the constant term of h' is zero. Since x_s is not a product of two non-units, $g(x_s) \neq x_s(1 + h')$ unless $h' = 0$. Thus $g(x_s) = x_s$ for all s and g is the identity.

(2) Left to the reader.

(3) By part (1), $\deg g \neq (-\infty, -\infty)$. Let $\deg g = (a, s)$. Since g is unipotent,

$$g(x_i) = x_i(1 + h') + h_{t_i} + \text{higher degree terms},$$

where h_{t_i} is the nonzero component of lowest degree that does not involve x_i. By definition, $t_s = a$ and if $h_{t_i} \neq 0$, then $t_i \geq a$ for all i. Note that h_{t_s} is a linear combination of certain monomials $x_1^{d_1} \cdots \widehat{x}_s \cdots x_n^{d_n}$. We claim that each $(d_1, \dots, \widehat{d_s}, \dots, d_n)$ is in $(T_s)_{\geq 2}$.

Applying g to the equation $x_i x_s = p_{si} x_s x_i$ for each i and removing all terms with x_s, we obtain that

$$x_i h_{t_s} + \text{higher degree terms} = p_{si} h_{t_s} x_i + \text{higher degree terms}.$$

For any nonzero monomial component $c x_1^{d_1} \cdots \widehat{x}_s \cdots x_n^{d_n}$ of h_{t_s}, the above equation yields

$$\prod_{j=1, j\neq s}^{n} p_{ji}^{d_j} = p_{si},$$

which is the equation defining T_s. Note that $t_s = \sum_j d_j \geq 2$, so $(d_1, \ldots, \widehat{d}_s, \ldots, d_n)$ is in $(T_s)_{\geq 2}$. The claim is proved.

Let $F = -h_{t_s}$, which is a linear combination of elements of $f_d \in X^{(T_s)_{\geq 2}}$ of total degree t_s, and then let $g' = g(F, s)g$. One can show that, for any $i \neq s$, $h'_{t_i} = h_{t_i}$ and $h'_{t_s} = h_{t_s} - F = 0$. By definition, $\deg g' > \deg g = \deg g(F, s)$. $\square$

Theorem 3.4. *Let $A = k_{p_{ij}}[\underline{x}_n]$ be a skew polynomial ring satisfying* (H1). *The following are equivalent.*

(1) *Every automorphism of A is affine. Equivalently,* $\mathrm{Aut}_{\mathrm{uni}}(A)$ *is trivial.*
(2) *For any commutative domain C containing k, every k-algebra automorphism of $A \otimes C$ is C-affine.*
(3) $(T_s)_{\geq 2} = \emptyset$ *for all s.*

If, in addition, $\mathbb{Z} \subset k$, then (1)–(3) *are also equivalent to the next two.*

(4) *Every locally nilpotent derivation of A of nonzero degree is zero.*
(5) *For any commutative domain C containing k, every locally nilpotent derivation of $A \otimes C$ of nonzero degree* (*with respect to the x_i-grading*) *is zero when restricted to A.*

Proof. (2) $\Rightarrow$ (1) Trivial.

(1) $\Rightarrow$ (3) Suppose that $(T_s)_{\geq 2}$ is non-empty for some s. Then the system (2.9.1) has a solution

$$(d_1, d_2, \ldots, d_{s-1}, d_{s+1}, \ldots, d_n),$$

where $d_i \geq 0$ and $\sum_i d_i \geq 2$. Let $f = x_1^{d_1} \cdots x_{s-1}^{d_{s-1}} x_{s+1}^{d_{s+1}} \cdots x_n^{d_n}$; this has degree at least 2. Then, by (2.10.1), the map

$$g : x_i \to \begin{cases} x_i & \text{if } i \neq s, \\ x_s + f & \text{if } i = s \end{cases}$$

extends to a non-affine algebra automorphism of A.

(3) ⇒ (2) Let $\mathfrak{m}$ be the graded ideal $A_{\geq 1} \otimes C$. Suppose that g is a non-C-affine automorphism of $A \otimes C$. Since each x_i is not central, each $g(x_i)$ has zero constant term (Lemma 3.2). Consequently, $g(x_i) \in \mathfrak{m}$. Thus g preserves the ideal $\mathfrak{m}$. Using the $\mathfrak{m}$-adic filtration, $\operatorname{gr} g$ is a C-affine automorphism of $\operatorname{gr} A \otimes C$, which is isomorphic to $A \otimes C$. Hence $h := g(\operatorname{gr} g)^{-1}$ is an algebra automorphism of $A \otimes C$ such that $h|_C = Id_C$, and $h(x_i) = x_i +$ higher degree terms for all i. That is, h is a unipotent automorphism of the C-algebra $A \otimes C$. Since g is not C-affine, neither is h. The assertion follows from Lemma 3.3(3) (when working with the base commutative ring C).

(5) ⇒ (4) Trivial.

(4) ⇒ (3) Suppose that, for some s, $(T_s)_{\geq 2}$ is non-empty, containing some element $(d_1, \ldots, \widehat{d_s}, \ldots, d_n)$. Let $f = x_1^{d_1} \cdots x_{s-1}^{d_{s-1}} x_{s+1}^{d_{s+1}} \cdots x_x^{d_n}$. Since this has degree at least 2, the map (2.10.2)

$$\partial : x_i \mapsto \begin{cases} 0 & \text{if } i \neq s, \\ f & \text{if } i = s \end{cases}$$

extends to a locally nilpotent derivation of degree at least 2.

(2) ⇒ (5) Here we need the hypothesis that $\mathbb{Z} \subset k$. After localizing, we may assume that k is a field of characteristic zero.

Let ∂ be a nonzero locally nilpotent derivation of $A \otimes C$. Let $g_c := \exp(c\partial)$ for $c \in k$. We know that the constant term of $g_c(x_i)$ is zero for all i and c. Then the constant term of $\partial^n(x_i)$ is zero for all n. If the degree of ∂ is not zero, then g_c is not C-affine, a contradiction. □

An immediate consequence of Lemma 2.9 and Theorem 3.4 is: if $C(k_{p_{ij}}[\underline{x}_n^{\pm 1}]) \subset k\langle x_1^{\pm\alpha_1}, \ldots, x_n^{\pm\alpha_n}\rangle$ for some $\alpha_1, \ldots, \alpha_n \geq 2$, then $\operatorname{Aut}_{\text{uni}}(A)$ is trivial.

The following is easy to check.

Lemma 3.5. *Assume* (H2). *Then $T_s = \emptyset$ if and only if T_s is finite if and only if $(T_s)_{\geq 2} = \emptyset$.*

The next theorem is a version of Theorem 3.4 when (H1) is replaced by (H2). Note that this is part of Theorem 3.1. Its proof is similar to the proof Theorem 3.4 and therefore is omitted. Let $\operatorname{Aut}_{\text{uni-}C}(A \otimes C)$ be the set of k-algebra automorphisms g of $A \otimes C$ such that $g|_C = Id_C$ and $g(x_i) = x_i +$ higher degree terms for all i.

Theorem 3.6. *Let $A = k_{p_{ij}}[\underline{x}_n]$ be a skew polynomial ring satisfying* (H2). *The following are equivalent.*

(1) $\operatorname{Aut}_{\text{uni}}(A) = \{1\}$
(2) *For any commutative domain C containing k,* $\operatorname{Aut}_{\text{uni-}C}(A \otimes C) = \{1\}$.
(3) $T_s = \emptyset$ *for all s.*

If $\mathbb{Z} \subset k$, then the above is also equivalent to

(4) $\mathrm{LNDer}(A) = \{0\}$.
(5) *For any commutative domain C containing k, every locally nilpotent derivation ∂ of $A \otimes C$ with $\partial|_C = 0$ is zero.*

By Theorem 3.4 or Theorem 3.6, if A is the algebra in Example 2.4(4), then it is easy to check that each $T_s = \emptyset$, so $\mathrm{Aut}(A)$ is affine. (Alternatively, one can apply Lemma 2.9.) Here is another example.

Example 3.7. Let $n = 4$ and $i^2 = -1$. Let

$$p_{12} = i, \quad p_{13} = i, \quad p_{14} = i, \quad p_{23} = -i, \quad p_{24} = i, \quad p_{34} = 1.$$

Then $A := k_{p_{ij}}[x_1, x_2, x_3, x_4]$ is a PI algebra with its center generated by x_i^4, $x_1^2x_2^2x_3^2$, $x_1^2x_2^2x_4^2$ and $x_3^2x_4^2$. Therefore $C(A)$ is not isomorphic to the polynomial ring; in fact, the center is not Gorenstein. One can check directly that $C(k_{p_{ij}}[\underline{x}_n^{\pm 1}]) \subset k\langle x_1^{\pm 2}, \ldots, x_4^{\pm 2}\rangle$. Therefore $\mathrm{Aut}(A)$ is affine by Lemma 2.9 and Theorem 3.4.

Along these lines, here is another part of Theorem 3.1.

Theorem 3.8. *Let $A = k_{p_{ij}}[\underline{x}_n]$ be a skew polynomial ring satisfying* (H2). *The following are equivalent.*

(1) $\mathrm{Aut}(A)$ *is affine.*
(2) $C(k_{p_{ij}}[\underline{x}_n^{\pm 1}]) \subset k\langle x_1^{\pm \alpha_1}, \ldots, x_n^{\pm \alpha_n}\rangle$ *for some $\alpha_1, \ldots, \alpha_n \geq 2$.*
(3) $C(k_{p_{ij}}[\underline{x}_n]) \subset k\langle x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\rangle$ *for some $\alpha_1, \ldots, \alpha_n \geq 2$.*
(4) $T_s = \emptyset$ *for all $s = 1, \ldots, n$.*

Proof. (1) $\Rightarrow$ (4) If $T_s \neq \emptyset$, then $(T_s)_{\geq 2} \neq \emptyset$. Hence, picking some element in $(T_s)_{\geq 2}$, the construction (2.10.1) defines a non-affine automorphism of A.

(4) $\Rightarrow$ (2) Let b_s be the smallest positive integer such that $x_s^{b_s}$ is in the center of A (and in the center of $k_{p_{ij}}[\underline{x}_n^{\pm 1}]$). Since all p_{ij} are roots of unity, b_s exists for each s.

For each s, let a_s be the smallest positive integer such that $x_1^{a_1} \cdots x_s^{a_s} \cdots x_n^{a_n} \in C(k_{p_{ij}}[\underline{x}_n^{\pm 1}])$ for some a_i. Then every monomial in the center is of the form $x_1^{c_1} \cdots x_s^{c_s} \cdots x_n^{c_n}$, where $a_s \mid c_s$. Suppose the assertion in (2) fails. Then $a_s = 1$ for some s. By multiplying by $x_i^{-b_i}$ if necessary, we may assume that there are $a_i > 1$ for all $i \neq s$ such that $x^{-a_1} \cdots x_s \cdots x_n^{-a_n} \in C(k_{p_{ij}}[\underline{x}_n^{\pm 1}])$. Equivalently, $x^{a_1} \cdots x_s^{-1} \cdots x_n^{a_n} \in C(k_{p_{ij}}[\underline{x}_n^{\pm 1}])$. Thus $T_s \neq \emptyset$ by Lemma 2.9(2).

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) Follows by Lemma 2.9(2).

(4) $\Rightarrow$ (1) From the earlier parts, we know that (4) $\Rightarrow$ (3); therefore no x_i is central in A: A satisfies (H1). The proof now follows by Lemma 3.5 and Theorem 3.4. $\square$

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. The equivalences of (1)–(4) are given in Theorem 3.8.

(4) $\Rightarrow$ (5) This is Theorem 2.11(3).

(5) $\Rightarrow$ (6) Trivial.

(6) $\Rightarrow$ (1) This is Theorem 1.13(1).

(7) $\Leftrightarrow$ (4) is given in Theorem 3.6. $\square$

The next proposition takes care of $\mathrm{Aut}_{\mathrm{gr}}(A)$ in many cases.

Proposition 3.9. *Suppose that $p_{ij} \neq 1$ for all $i < j$. Let*

$$S = \{\sigma \in S_n \mid p_{ij} = p_{\sigma(i)\sigma(j)} \quad \textit{for all } i, j\}.$$

(1) *Then*

$$\mathrm{Aut}_{\mathrm{gr}}(A) = S \ltimes (k^\times)^n.$$

(2) *Suppose the conditions in Theorem 3.4(1)–(3) hold. Then*

$$\mathrm{Aut}(A) = \mathrm{Aut}_{\mathrm{gr}}(A) = S \ltimes (k^\times)^n.$$

If, further, $\mathbb{Z} \subset k$, then every locally nilpotent derivation of A is zero.

Proof. (1) It is clear that $S \ltimes (k^\times)^n \subset \mathrm{Aut}_{\mathrm{gr}}(A)$. We claim that $S \ltimes (k^\times)^n \supset \mathrm{Aut}_{\mathrm{gr}}(A)$. Since $p_{ij} \neq 1$, every graded automorphism g of A is of the form $g : x_i \to c_i x_{\sigma(i)}$ for some $c_i \in k^\times$ and $\sigma \in S_n$ [13, Lemma 2.5(e)]. Then $\sigma \in S$. The claim is proved.

(2) $\mathrm{Aut}_{\mathrm{uni}}(A)$ is trivial by assumption, so $\mathrm{Aut}(A) = \mathrm{Aut}_{\mathrm{gr}}(A)$ by Lemma 3.2.

The assertion about locally nilpotent derivations follows from a similar argument in the proof of (2) $\Rightarrow$ (5) in Theorem 3.4. $\square$

In the following special case, $\mathrm{Aut}(A)$ being affine is equivalent to $C(A)$ being isomorphic to a polynomial ring.

Theorem 3.10. *Let $A = k_{p_{ij}}[\underline{x}_n]$ be a skew polynomial ring satisfying* (H1) *and* (H2). *Suppose that the subgroup of $k^\times$ generated by parameters $\{p_{ij} \mid i < j\}$ is equal to $\langle q \rangle$ where ℓ is prime and q is a primitive ℓth root of unity. Then the following are equivalent.*

(1) $\mathrm{Aut}(A)$ *is affine.*
(2) *For any commutative domain C which is k-flat, every k-algebra automorphism of $A \otimes C$ is C-affine.*
(3) $(k^\times)^n \subset \mathrm{Aut}(A) \subset S_n \ltimes (k^\times)^n$, *where $(k^\times)^n$ is viewed as* $\mathrm{Aut}_{\mathbb{Z}^n-\mathrm{gr}}(A)$.
(4) $C(A)$ *is isomorphic to a polynomial ring.*

(5) *A is a free module over $C(A)$.*
(6) *The determinant* $\det(\phi_{ij})_{n\times n}$ *is invertible in* $\mathbb{Z}/(\ell)$*, where* ϕ_{ij} *are determined by* $p_{ij} = q^{\phi_{ij}}$ *for all i and j – see* (H2').
(7) *$d_w(A/C(A))$ is dominating where* $w = \mathrm{rk}(A/C(A))$*.*

If $\mathbb{Z} \subset k$, then the above are also equivalent to

(8) $\mathrm{Aut}(A[t]) = \mathrm{Aut}_{\mathrm{tr}}(A[t])$*.*
(9) *Every locally nilpotent derivation of A is zero.*

Proof. The equivalence of (1), (7) and (9) is given in Theorem 3.1.

(1) ⇒ (3) If $\mathrm{Aut}(A)$ is affine, then $\mathrm{Aut}(A) = \mathrm{Aut}_{\mathrm{gr}}(A)$. The assertion follows from Proposition 3.9(1).

(3) ⇒ (1) Part (3) says that there are no non-trivial unipotent automorphisms. Hence every automorphism is affine by Lemma 3.2(4).

(1) ⇒ (4) If $C(A)$ is not a polynomial ring, by Lemma 2.5, there is a solution $(d_1, \ldots, d_n) \in \mathbb{N}^n$ to the system of equations

$$\prod_{j=1}^{n} p_{ij}^{d_j} = 1, \quad \text{for all } i$$

with $d_s = 1$. So for each i,

$$p_{is} = p_{si}^{-1} = \prod_{j=1, j\neq s}^{n} p_{ij}^{-d_j} = \prod_{j=1, j\neq s}^{n} p_{ij}^{-d_j + a\ell} \quad \text{for any } a \in \mathbb{N}.$$

Hence, for some $a > 0$, $(-d_1 + a\ell, \cdots, -d_{s-1} + a\ell, -d_{s+1} + a\ell, \cdots, -d_n + a\ell) \in \mathbb{N}^{n-1}$ is a solution to (2.9.1) with $\sum_{j\neq s}(-d_j + a\ell) \geq 2$. Thus $\mathrm{Aut}(A)$ is not affine, a contradiction. Therefore $C(A)$ is a polynomial ring.

(4) ⇔ (5) Lemma 2.3.

(4) ⇒ (6) follows from Lemma 2.5 and linear algebra.

(6) ⇒ (4) Example 2.4(1).

(5) ⇒ (7) Proposition 2.8.

(7) ⇒ (2) [6, Lemma 3.2(1)].

(2) ⇒ (1) is obvious.

(7) ⇒ (8) Theorem 1.13.

(8) ⇒ (9) [6, Lemma 3.3]. □

Note that part (1) does not imply part (4) if ℓ is 4 (which is not prime) – see Example 3.7.

Here are some cases in which the hypotheses of Proposition 3.9 hold.

(1) Assume $p_{ij} = q$ for all $i < j$ and q is not a root of unity. For any fixed s between 1 and n, the condition (2.9.1) says, in this case, that for any $i < s$,

$$\sum_{j<i} d_j - \sum_{j>i, j\neq s} d_j = -1 \tag{3.10.1}$$

and, for any $i > s$,

$$\sum_{j<i, j\neq s} d_j - \sum_{j>i} d_j = 1 \tag{3.10.2}$$

for non-negative integers $(d_1, \dots, \widehat{d_s}, \dots, d_n)$. If $n = 2$, it is easy to check that there is no solution. If $n = 3$, there is one solution when $s = 2$, which is $(d_1, d_3) = (1, 1)$. As in [2, Theorem 1.4.6(i)], we have

$$\begin{aligned} &\mathrm{Aut}(k_q[x_1, x_2, x_3]) \\ &\quad = \{g : x_1 \mapsto a_1 x_1,\ x_2 \mapsto a_2 x_2 + b x_1 x_3,\ x_3 \mapsto a_3 x_3,\ \text{where } a_i \in k^\times,\ b \in k\}. \end{aligned}$$

Assume now $n \geq 4$. When $s = 1$, taking $i = 2$, (3.10.2) becomes $-\sum_{j>2} d_j = 1$, which has no solution. Similarly, (3.10.1) has no solution for $s = n$. When $1 < s < n$, take $i = 1$ and $i = n$; then (3.10.1) and (3.10.2) (and the condition that $\sum_j d_j \geq 2$) imply that $d_1 = d_n = 1$ and $d_j = 0$ for all $1 < j < n$, $j \neq s$. Since $n \geq 4$, there is another i with $i \neq 1, s, n$. Then either (3.10.1) or (3.10.2) gives a contradiction. In summary, we recover [2, Theorem 1.4.6(ii)], which states that $\mathrm{Aut}(k_q[x_1, \dots, x_n]) = (k^\times)^n$ if and only if $n = 2$ or $n \geq 4$.

(2) If $p_{ij} = q$ for all $i < j$ and q is a root of unity, then Example 1.8 shows that $\mathrm{Aut}(A)$ is not affine when n is odd. But one can check by using Proposition 3.9 that if n is even, then $\mathrm{Aut}(A)$ is affine. We will give another proof of this fact later.

Theorem 3.11. *Let s_0 be an integer between 1 and n. Suppose that T_s is finite for all $s \neq s_0$. Then every unipotent automorphism g is a product of elementary automorphisms:*

$$g = g(F_1, s_{n_1}) g(F_2, s_{n_2}) \cdots g(F_w, s_{n_w}).$$

Moreover, we may choose that the degrees $\deg g(F_i, s_{n_i})$ are strictly increasing, or alternatively, strictly decreasing. In either case, the decomposition is unique.

Proof. We will construct the factorization and show that the degrees $\deg g(F_i, s_{n_i})$ are strictly increasing. Replacing g by g^{-1}, we obtain the case when the degrees are strictly decreasing.

We use downward induction on $\deg g$. By the hypothesis that every T_s except possibly T_{s_0} is finite, we first assume that $\deg g = (a, s)$ and $a > \sum_j d_j$ for any $(d_1, \dots, \widehat{d_s}, \dots, d_n) \in \bigcup_{s \neq s_0} T_s$. For any $i \neq s_0$, if

$$g(x_i) = x_i(1 + h') + h_{t_i} + (\text{higher degree terms}),$$

then the proof of Lemma 3.3(3) shows that $h_{t_i} = 0$ and $g(x_i) = x_i(1 + h')$. Since x_i is not a product of two non-units, $g(x_i) = x_i$ for all $i \neq s_0$.

Now let $g(x_{s_0}) = x_{s_0} +$ (higher degree terms) and write $g(x_{s_0}) = \sum f_i x_{s_0}^i$ and $g^{-1}(x_{s_0}) = \sum_j h_j x_{s_0}^j$ with $f_i, h_j \in k_{p_{ij}}[x_1, \ldots, \widehat{x}_{s_0}, \ldots, x_n]$. Suppose m, q are maximal so that $f_m h_q \neq 0$. Then

$$x_{s_0} = gg^{-1}(x_{s_0}) = h_q f_m^q x_{s_0}^{mq} + (\text{lower degree terms}).$$

Thus $m = q = 1$ and $h_1 \in k^\times$. Similarly, $f_1 \in k^\times$. Since g and g^{-1} are unipotent, $f_1 = h_1 = 1$. This means that $g(x_{s_0}) = x_{s_0} + \sum_f c_f f$, where f ranges over a set of monomials in $k_{p_{ij}}[x_1, \ldots, \widehat{x}_{s_0}, \ldots, x_n]$. Now the argument in the proof of Lemma 3.3(3) shows that $f \in T_{s_0}$ for all f. In this case, it is easy to see that g can be decomposed into the form as given,

$$g = g(F_1, s_0)g(F_2, s_0) \cdots g(F_w, s_0),$$

and these $g(F_i, s_0)$ commute. Uniqueness follows from the fact that each F_i is precisely a homogeneous component of f.

Next assume that g is not the identity. By Lemma 3.3(3), $\deg g(F, s)g > \deg g$ for some F and s. By the inductive hypothesis, $g(F, s)g = g(F_2, s_2) \cdots g(F_w, s_w)$. Then $g = g(-F, s)g(F_2, s_2) \cdots g(F_w, s_w)$. Let $F_1 = -F$ and $s_1 = s$. The uniqueness of (F_1, s_1) can be read off from the proof of Lemma 3.3(3) and the fact that $\deg g(F_i, s_i)$ are increasing. The inductive hypothesis also says that the (F_i, s_i) are unique for $i > 1$. The assertion now follows. $\square$

Proof of Theorem 4. Let g be in $\operatorname{Aut}(A)$. Since A satisfies (H1), $g(x_i)$ has no constant term by Lemma 3.2. Then the associated graded map $\operatorname{gr} g$ is a graded (hence affine) automorphism of A. Now $g(\operatorname{gr} g)^{-1}$ is a unipotent automorphism. The assertion follows from the equation $g = [g(\operatorname{gr} g)^{-1}](\operatorname{gr} g)$ and Theorem 3.11. $\square$

To conclude this section we give some examples.

Example 3.12.

(1) Let q be a primitive ℓth root of unity and $\ell = abc$, where $a, b, c \geq 2$ are pairwise coprime. If $p_{12} = q^a$, $p_{13} = q^b$ and $p_{23} = q^c$ and $A = k_{p_{ij}}[x_1, x_2, x_3]$, then one can check that $T_s = \emptyset$ for $s = 1, 2, 3$ and $\operatorname{Aut}(A) = (k^\times)^3$. Similar statements can be made if there are more than three generators.

(2) If $A = k_{p_{ij}}[x_1, x_2, x_3]$ is not PI, then it is easy to check that each T_s is finite for $s = 1, 2, 3$. As a consequence of Theorem 3.11, $\operatorname{Aut}(A)$ is tame. Here is an explicit example. Assume that q is not a root of unity. Let $p_{12} = q^m$, $p_{13} = q$ and $p_{23} = q^n$

for some integers $m, n \geq 1$. Then $T_1 = \emptyset$, $T_2 = \{(n, \widehat{d_2}, m)\}$ and $T_3 = \emptyset$. Hence every automorphism of $k_{p_{ij}}[x_1, x_2, x_3]$ is of the form

$$x_1 \mapsto a_1 x_1,$$
$$x_2 \mapsto a_2 x_2 + b x_1^n x_3^m,$$
$$x_3 \mapsto a_3 x_3,$$

where $a_i \in k^\times$ and $b \in k$ (Theorem 3.11). This should be compared with [2, Theorem 1.4.6(i)].

Example 3.13.

(1) (See [6, Example 3.8].) If $p_{12} = 1$, $p_{13} = q$, $p_{23} = q$, where q is not a root of unity, then the system of equations (2.9.1) for $s = 1$ and for $s = 2$ has only one solution $(d_2, d_3) = (1, 0)$, and the system of equations for $s = 3$ has no solution. Therefore these systems of equations have no solution with $\sum_j d_j \geq 2$. By Theorem 3.4, every automorphism of $A = k_{p_{ij}}[x_1, x_2, x_3]$ is affine.

(2) By the analysis of the case $n = 2$, every automorphism of $B = k_p[x_4, x_5]$ (when $p \neq 1$ and $p^w = 1$) is affine.

(3) The tensor product $C = A \otimes B$ is a skew polynomial ring $k_{p_{ij}}[x_1, \ldots, x_5]$. But C has a non-affine automorphism determined by

$$g(x_1) = x_1 + x_2 x_4^w x_5^w,$$
$$g(x_i) = x_i, \quad \text{for all } i > 1.$$

So even if A and B only have affine automorphisms, $A \otimes B$ may have non-affine automorphisms. Compare this with [6, Theorem 5.5].

4. Miscellaneous operations and constructions

In this section we discuss some general methods that deal with automorphisms and discriminants, for use in proving Theorem 2. Two examples: in Subsection 4.1 we develop tools to study automorphisms of tensor products of algebras. In Subsection 4.4 we look at filtered algebras: if B is filtered and C is a central subalgebra of B; then with some extra hypotheses, $\operatorname{gr} d_w(B/C) = d_w(\operatorname{gr} B / \operatorname{gr} C)$ (Proposition 4.10).

4.1. Tensor products and the categories A_{-s} *and* Af_{-s}

Recall from [6, Definition 2.4] that the category Af consists of all k-flat algebras A satisfying the following conditions:

(1) A is an algebra with standard filtration such that the associated graded ring $\operatorname{gr} A$ is a connected graded domain,
(2) A is a finitely generated free module over its center $C(A)$, and
(3) the discriminant $d(A/C(A))$ is dominating.

The morphisms in this category are isomorphisms of algebras.

We extend this definition to a more general situation.

Definition 4.1. Let s be a non-negative integer.

(1) Let A_{-s} be the category consists of all k-flat algebras A satisfying the following conditions:
   (a) A is an algebra with standard filtration such that the associated graded ring $\operatorname{gr} A$ is a connected graded domain,
   (b) the w-discriminant $d_w(A/C(A))$ is $(-s)$-dominating where w is the rank $\operatorname{rk}(A/C(A))$.
(2) Let Af_{-s} be the category consists of all k-flat algebras A satisfying the following conditions:
   (a) A is in A_{-s}, and
   (b) A is a finitely generated free module over its center.

The morphisms in these categories are isomorphisms of algebras.

Remark 4.2.

(1) $\mathrm{Af} = \mathrm{Af}_0$.
(2) Af_{-s} is a full subcategory of A_{-s} for any s.
(3) If A is in A_{-s}, then every automorphism of A is $(-s)$-affine; see the proof of [6, Lemma 2.6].

Let A be a noncommutative domain. Let $\mathcal{D} := \{d_i\}_{i\in I}$ be a subset of A with gcd y. Let $\mathcal{D}^n$ denote $\{d_{i_1} \cdots d_{i_n} \mid d_{i_s} \in \mathcal{D}\} \subseteq A$. Let A' be another domain. We say $\mathcal{D}$ is *A'-saturated* if for every positive integer n and every $0 \neq f \in A'$, the subset $\mathcal{D}^n \otimes f$ in $A \otimes A'$ has gcd $y^n \otimes f$ (also written as $y^n f$). If $\mathcal{D}$ is a subset of monomials in $k_{p_{ij}}[\underline{x}_n]$, then $\mathcal{D}$ is A'-saturated for any domain A'.

Lemma 4.3. *Let A and A' be two domains with generating sets X and X' and with semi-bases b and b' over their centers $C(A)$ and $C(A')$. Suppose that $C(A)$ and $C(A')$ are k-flat and that b' is a quasi-basis. Let $m = \operatorname{rk}(A/C(A))$ and $n = \operatorname{rk}(A'/C(A'))$. Let $w = mn$.*

(1) *The discriminant $d_w(A \otimes A'/C(A \otimes A'))$ is the gcd of elements in $\mathcal{D}(X/b)^n \otimes \mathcal{D}(X'/b')^m$.*

(2) *Suppose that A' is free over $C(A')$ and that $\mathcal{D}(X/b)$ is A'-saturated. Then $d_w(A \otimes A'/C(A \otimes A')) = d_m(A/C(A))^n d_n(A'/C(A'))^m$.*

Proof. (1) Since b and b' are semi-bases of A and A' respectively, $b \otimes b'$ is a semi-basis of $A \otimes A'$ over its center. Also $X \otimes X'$ is a generating set of $A \otimes A'$ over its center. For each subset $Z := \{x_{i_s} \otimes x'_{j_s}\}_{s=1}^{w}$ of $X \otimes X'$ with $\det(Z : b \otimes b') \neq 0$, since $b' = \{b'_1, \ldots, b'_n\}$ is a quasi-basis, one can rewrite Z as

$$Z = \{x_{i_{s,t}} \otimes c_{s,t} b'_t \mid 1 \leq s \leq m, 1 \leq t \leq n\},$$

where $c_{s,t} \in C'_t$ (where C'_t is the set of nonzero elements of the form $(x' : b'_t)$, as in Definition 1.10). Let $\widehat{Z}$ be the subset

$$\{x_{i_{s,t}} \otimes b'_t \mid 1 \leq s \leq m, 1 \leq t \leq n\}.$$

Then $\det(Z : \widehat{Z}) = \prod_{s,t} c_{s,t}$, which is in $[(C'_1) \cdots (C'_n)]^m = (X'/b')^m$. By linear algebra,

$$\det(\widehat{Z} : b \otimes b') = \pm \prod_{t=1}^{n} \det(\{x_{1,t}, \ldots, x_{m,t}\} : b) \in (X/b)^n.$$

By the proof of [6, Lemma 5.3], $d_w(b \otimes b' : tr) = d_m(b : \mathrm{tr})^n d_n(b' : \mathrm{tr})^m$. For any two subsets Z_1 and Z_2 of $X \otimes X'$, we have

$$d_w(Z_1, Z_2 : \mathrm{tr}) = \pm \alpha_1 \alpha_2 \beta_1 \beta_2 d_w(b \otimes b' : \mathrm{tr}),$$

where α_1 and α_2 are the product of n elements of the form $\det(\{x_{i_1}, \ldots, x_{i_m}\} : b)$ and β_1 and β_2 are the product of m elements of the form $\prod_{s=1}^{n} c_s$. Therefore $d_w(Z, \widehat{Z} : \mathrm{tr})$ is in $\mathcal{D}(X/b)^n \otimes \mathcal{D}(X'/b')^m$. The assertion follows.

(2) If A' is free over $C(A')$, we take $X' = b'$ to be a basis of A'. In this case, $\mathcal{D}(X'/b')$ is a singleton $\{y\}$, where $y = d_n(A'/C(A'))$. By part (1), the w-discriminant of $A \otimes A'$ over its center is the gcd of $\mathcal{D}(X/b)^n \otimes y^m$. The assertion follows from the A'-saturatedness of $\mathcal{D}(X/b)$. □

Lemma 4.4. *Let s and t be non-negative integers. Assume that A and B are k-flat filtered algebras such that $\mathrm{gr}\, A \otimes \mathrm{gr}\, B$ is a connected graded domain. In part (4) we also assume that the center of A is k-flat.*

(1) *If $A \in \mathrm{Af}_{-s}$ and $B \in \mathrm{Af}_{-t}$, then $A \otimes B \in \mathrm{Af}_{-(s+t)}$.*
(2) *If $A \in \mathrm{A}_{-s}$, then $A[t] \in \mathrm{A}_{-(s+1)}$.*
(3) *If $A \in \mathrm{Af}_{-s}$, then $A[t] \in \mathrm{Af}_{-(s+1)}$.*
(4) *Suppose A is in A_{-s} and B is in Af_{-t}. If there is a generating set X of A containing a semi-basis b such that $\mathcal{D}(X/b)$ is B-saturated, then $A \otimes B$ is in $\mathrm{A}_{-(s+t)}$.*

Proof. (1) By hypothesis, $\operatorname{gr}(A \otimes B) \cong \operatorname{gr} A \otimes \operatorname{gr} B$ is a connected graded domain. It is clear that $A \otimes B$ is finitely generated free over its center $C(A) \otimes C(B)$. It remains to show that the discriminant is $(-(s+t))$-dominating. By [6, Lemma 5.3], $d(A \otimes B/C(A \otimes B)) = d(A/C(A))^n d(B/C(B))^m$, where $m = \operatorname{rk}(A/C(A))$ and $n = \operatorname{rk}(B/C(B))$. When $d(A/C(A))$ is $(-s)$-dominating and $d(B/C(B))$ is $(-t)$-dominating, it follows from the definition that $d(A/C(A))^n d(B/C(B))^m$ is $(-(s+t))$-dominating.

(2) Let $Y = \bigoplus_{i=1}^n kx_i$ be the generating space of A as in Definition 1.6. Then $Y' = Y \oplus kt$ is a generating space of $A[t]$. By Lemma 1.12(2), $d_w(A[t]/C(A[t])) = d_w(A/C(A))$. If $d_w(A/C(A))$ is $(-s)$-dominating with respect to Y, then it is $(-(s+1))$-dominating with respect to Y'.

(3) This is a special case of (2).

(4) By Lemma 4.3(2), $d_w(A \otimes B/C(A \otimes B)) = d_m(A/C(A))^n d_n(B/C(B))^m$. Then the proof of part (1) works. □

One immediate application of Lemma 4.4(4) is the following: assume $A := k_{p_{ij}}[\underline{x}_n]$ satisfies (H2). Suppose that $C(A) \subset k\langle x_1^{a_1}, \ldots, x_n^{a_n}\rangle$, where $a_i \geq 2$ for all i. By Theorem 3.1, A is in A_0. Let B be the algebra $k\langle x,y\rangle/(y^2x - xy^2, yx^2 + x^2y)$ given in [6, Example 5.1], which is in $\mathrm{Af} = \mathrm{Af}_0$. By Lemma 4.4(4), $A \otimes B$ is in A_0. Then $\operatorname{Aut}(A \otimes B)$ is affine by Theorem 1.13.

We generalize the notion of elementary automorphisms of skew polynomial rings, as in the introduction and (2.10.1), as follows. Suppose that $Y := \bigoplus_{i=1}^n kx_i$ generates A and let $g \in \operatorname{Aut}(A)$. We say that g is *elementary* if there is an s such that $g(x_i) = x_i$ for all $i \neq s$. In general elementary automorphisms are relatively easy to determine when all relations of A are understood.

An automorphism $g \in \operatorname{Aut}(A)$ is called *tame* if it is generated by affine and elementary automorphisms, and a subgroup G of $\operatorname{Aut}(A)$ is *tame* if every g in G is tame. Let A be a connected graded algebra. Recall that $g \in \operatorname{Aut}(A)$ is called unipotent if $g(x) - x$ is a linear combination of homogeneous elements of degree at least 2 for all $x \in A_1$.

Proposition 4.5. *Suppose that A is a graded domain generated by $Y := \bigoplus_{i=1}^n kx_i$ in degree 1 and that A is in A_{-1}. Then every unipotent automorphism is elementary. If, further, for every automorphism $h \in \operatorname{Aut}(A)$, $h(x_i)$ has no constant term, then* $\operatorname{Aut}(A)$ *is tame.*

Proof. Let g be a unipotent automorphism and write $g(x_i) = x_i + f_i$, where f_i is a linear combination of homogeneous elements of degree at least 2. Since the discriminant is (-1)-dominating, g is (-1)-affine by Remark 4.2(3). Hence $\deg g(x_i) \leq 1$ for all but one i. Thus $g(x_i) = x_i$ for all but one i. Therefore g is elementary.

If $h(x_i)$ has no constant term, then $\operatorname{gr} h$ is a graded automorphism (hence affine) and $h(\operatorname{gr} h)^{-1}$ is a unipotent automorphism. The final assertion follows the equation $h = [h(\operatorname{gr} h)^{-1}](\operatorname{gr} h)$. □

A special case is when A is the algebra $k_{p_{ij}}[\underline{x}_n]$ that satisfies (H1) and (H2). By Lemma 3.2(4), every automorphism of A is generated by graded and unipotent automorphisms. If A is in A_{-1}, then $T_s = \emptyset$ except for one s (Theorem 2.11(2)). By the proof of Theorem 3.11, every unipotent automorphism is of the form (2.10.1). Applying the above to the algebra D in Example 1.3(4), we obtain that every automorphism of D is determined by

$$g(x_s) = \begin{cases} a_1 x_1 + \sum_{i,j} b_{ij} x_2^{2+4i} x_3^{2+4j}, & s = 1, \\ a_s x_s, & s = 2, 3, \end{cases}$$

where $a_s \in k^\times$ and $b_{ij} \in k$ for all s, i and j.

4.2. Mod-p reduction

In this subsection we introduce a general method that deals with automorphisms of certain non-PI algebras. Let K be a commutative domain. We write $\mathrm{Aut}_{\mathrm{af}}(A)$ for the set of affine automorphisms of an algebra A.

Lemma 4.6. *Let K be finitely generated over $\mathbb{Z}$. Suppose S is a filtered K-algebra such that $\mathrm{gr}\, S$ is locally finite and connected graded. Suppose that $\mathrm{gr}\, S$ is a free K-module, namely, each $(\mathrm{gr}\, S)_i$ is free over K.*

(1) *If, for every quotient field $F \cong K/\mathfrak{m}$, $\mathrm{Aut}(S \otimes_K F) = \mathrm{Aut}_{\mathrm{af}}(S \otimes_K F)$, then $\mathrm{Aut}(S) = \mathrm{Aut}_{\mathrm{af}}(S)$.*
(2) *If, for every quotient field $F \cong K/\mathfrak{m}$, $\mathrm{Aut}(S \otimes_K F[t]) = \mathrm{Aut}_{\mathrm{tr}}(S \otimes_K F[t])$, then $\mathrm{Aut}(S[t]) = \mathrm{Aut}_{\mathrm{tr}}(S[t])$.*
(3) *If, for every quotient field $F \cong K/\mathfrak{m}$, every locally nilpotent derivation of $S \otimes_K F$ is zero, then every locally nilpotent derivation of S is zero.*

Proof. For every quotient field $F \cong K/\mathfrak{m}$, $S \otimes_K F$ is filtered and $\mathrm{gr}(S \otimes_K F)$ is naturally isomorphic to $(\mathrm{gr}\, S) \otimes_K F$, so we identify these two algebras. Since K is finitely generated over $\mathbb{Z}$, F is a finite field.

Since $\mathrm{gr}\, S$ is free over K, there is a K-basis of $\mathrm{gr}\, S$, say,

$$\{1\} \cup \{x_i\} \cup \{\text{higher degree terms}\}, \tag{4.6.1}$$

where $\bigoplus_i kx_i$ generates S as an algebra. We use the same symbols for a K-basis of S by lifting, and also for an F-basis of $S \otimes_K F$ (as $S \otimes_K F$ is free over F), and for an F-basis of $(\mathrm{gr}\, S) \otimes_K F$.

(1) Proceed by contradiction and suppose there is a non-affine automorphism $g \in \mathrm{Aut}(S)$. Then we have

$$g(x_i) = a_i + \sum_{i'} b_{ii'} x_{i'} + \sum_j c_{ij} y_j,$$

where $a_i, b_{ii'}, c_{ij} \in K$, some $c_{i_0 j_0} \neq 0$, and y_j are basis elements in (4.6.1) with degree at least 2. Let K' be the localization $K[c_{i_0 j_0}^{-1}]$ and let F be a quotient field of K'. Since K' is finitely generated over $\mathbb{Z}$, F is a finite field. This implies that the composition $K \to K' \to F$ is surjective and F is a quotient field of K. Note that $g \otimes_K F$ is an automorphism of $S \otimes_K F$. Since $c_{i_0 j_0} \neq 0$ in F, $g \otimes_K F$ is not affine, contradicting hypothesis. Therefore the assertion follows.

(2) Proceed by contradiction and suppose there is a non-triangular automorphism $h \in \operatorname{Aut}(S[t])$. Then there is an i such that

$$h(x_i) = \sum_{j \geq 0} f_j t^j,$$

where $f_j \in S$ and $f_n \neq 0$ for some $n > 0$. Writing $\{z_s\}_s$ for the basis given in (4.6.1), write $f_n = \sum_s c_s z_s$ for some $c_s \neq 0$. Let K' be the localization $K[c_s^{-1}]$ and let F be a quotient field of K'. Since K' is finitely generated over $\mathbb{Z}$, F is a finite field. This implies that the composition $K \to K' \to F$ is surjective and F is a quotient field of K. Note that $h \otimes_K F$ is an automorphism of $S \otimes_K F[t]$. Since $c_s \neq 0$ in F, $h \otimes_K F$ is not triangular, contradicting hypothesis. Therefore the assertion follows.

(3) The proof is similar and omitted. □

4.3. Factor rings

In this subsection we assume that A is filtered algebra with filtration $\{F_i A\}_{i \geq 0}$ such that the associated graded algebra is a domain. Let $Y = \bigoplus_{i=1}^n k x_i$ be a submodule of $F_1 A$ such that $F_1 A = Y \oplus k$. Assume that A is finitely generated free over its center R. Let I be an ideal of R and let $\bar{\cdot}$ denote the factor map $R \to R/I =: \overline{R}$ and the factor map $A \to A/I =: \overline{A}$.

Proposition 4.7. *Retain the above notations. Suppose that*

(1) $Y \cong \overline{Y}$,
(2) *the center of* $\overline{A}$ *is* $\overline{R}$,
(3) *the associated graded ring* $\operatorname{gr} \overline{A}$ *is a domain.*

Then $\overline{A}$ *is finitely generated free over* $\overline{R} = C(\overline{A})$ *and* $d(\overline{A}/\overline{R}) = \overline{d(A/R)}$. *As a consequence, if* $d(A/R)$ *is* $(-s)$*-dominating, so is* $d(\overline{A}/\overline{R})$.

Proof. Since $\overline{A} \cong A \otimes_R \overline{R}$, $\overline{A}$ is finitely generated free over $\overline{R}$: we may use the R-free basis of A for the $\overline{R}$-free basis of $\overline{A}$. Then $\operatorname{tr}(\overline{f}) = \overline{\operatorname{tr}(f)}$ for all $f \in A$, and consequently $d(\overline{A}/\overline{R}) = \overline{d(A/R)}$. The last assertion follows from the fact $\operatorname{gr} \overline{A}$ is a domain. □

In general if $d(\overline{A}/\overline{R})$ is $(-s)$-dominating, $d(A/R)$ may not be $(-s)$-dominating. Consider the following example.

Example 4.8. Let A be the algebra $k\langle x, y\rangle/(y^2x - xy^2, yx^2 - x^2y)$. Then the center R of A is generated by x^2, y^2 and $z := xy + yx$, and the discriminant $d(A/C(A)) = (xy - yx)^4$. It is easy to check that $(xy - yx)^4$ is not dominating in A.

Let $\overline{A}$ be the algebra $A/(x^6 - y^2)$, which is studied in [6, Example 5.6]. By Proposition 4.7, $d(\overline{A}/\overline{R}) = \overline{d(A/R)} = (xy - yx)^4$ which can be written as $(z - 2x^4)^2(z + 2x^4)^2$ in $\overline{R}$. By the analysis in [6, Example 5.6] which uses a non-standard filtration determined by $\deg x = 1$ and $\deg y = 3$, $(z - 2x^4)^2(z + 2x^4)^2$ is dominating.

4.4. Discriminants of filtered algebras

Let Λ be a totally ordered abelian semigroup (e.g., $\mathbb{N}^n$ with the left lexicographic ordering). We say B is a Λ-filtered algebra if there is a filtration $F = \{F_g B \mid g \in \Lambda\}$ such that $B = \bigcup_{g\in\Lambda} F_g B$. The associated graded algebra is defined to be

$$\operatorname{gr}_F B = \bigoplus_{g\in\Lambda} F_g B / F_{<g} B,$$

where $F_{<g} = \sum_{h<g} F_h B$. For every nonzero $f \in B$, we can define the degree of f to be the degree of $\operatorname{gr} f$ in $\operatorname{gr}_F B$.

We do not assume that $\operatorname{gr} B$ is connected graded, even if $\Lambda = \mathbb{N}$. Inductively, we identify the k-module B_g with the graded k-module $\bigoplus_{h\le g}(\operatorname{gr} B)_h$ (with some choices) so that taking the principal term of f, denoted by $\operatorname{gr}(f)$, can be realized as a projection $B_g \to (\operatorname{gr} B)_h$ if $f \in B_h \setminus B_{<h}$. So B is identified with $\bigoplus_{g\in\Lambda}(\operatorname{gr} B)_g$ as a k-module, and we use $\xi : \operatorname{gr} B \to B$ denote the inverse of this identification map. By using ξ, elements in $\operatorname{gr} B$ can be viewed as elements in B. Two elements f and g in B or in $\operatorname{gr} B$ are said to be λ-*equivalent* if both $\deg f$ and $\deg g$ are no more than λ and $\deg(f - g) < \lambda$. In this case we write $f \equiv_\lambda g$.

Let C be the center (or more generally, a central subalgebra) of B such that B is finitely generated free over C with a basis $b = \{b_1 = 1, b_2, \cdots, b_w\}$. It is clear that $R := \operatorname{gr} C$ is a central subalgebra of $\operatorname{gr} B$. Let $\operatorname{gr} b$ denote the set $\{\operatorname{gr} b_1, \ldots, \operatorname{gr} b_w\}$. Suppose that

$$\operatorname{gr} B \text{ is finitely generated free over } \operatorname{gr} C \text{ with a basis } \operatorname{gr} b. \tag{4.8.1}$$

Note that in general, even if C is the center of B, (4.8.1) could fail. The following lemma is easy.

Lemma 4.9. *Assume (4.8.1) and let $\lambda, \lambda' \in \Lambda$. The following hold.*

(1) *If $\deg(f) = \lambda \ge \deg(g) = \lambda'$, then $\operatorname{gr}(fg) \equiv_{\lambda+\lambda'} \operatorname{gr}(f)\operatorname{gr}(g)$, $\operatorname{gr}(af) = a \operatorname{gr} f$ for $a \in k$, and $\operatorname{gr}(f + g) \equiv_\lambda \operatorname{gr}(f) + gr(g)$.*
(2) *If $\deg f \le \lambda$, then $\operatorname{tr}(\operatorname{gr} f) \equiv_\lambda \operatorname{gr}\operatorname{tr}(f)$.*

Proof. (1) Clear.

(2) It suffices to show the assertion when $\lambda = \deg f$. By (4.8.1), $fb_i = \sum_j r_{ij} b_j$ for some $r_{ij} \in R$ and $\deg r_{ij} b_j \leq \deg f b_i =: \phi$. Then

$$\operatorname{gr}(f)\operatorname{gr}(b_i) = \operatorname{gr}(fb_i) \equiv_\phi \sum_j \operatorname{gr}(r_{ij}b_j) \equiv_\phi \sum_j \operatorname{gr}(r_{ij})\operatorname{gr}(b_j)$$

with $\deg \operatorname{gr}(r_{ii}) \leq \phi - \deg(b_i) = \lambda$. Hence $\operatorname{tr}(\operatorname{gr}(f)) \equiv_\lambda \sum_i \operatorname{gr}(r_{ii})$. □

Proposition 4.10. *Retain the above notation and assume (4.8.1). If $d_w(\operatorname{gr} B/R)$ is nonzero, then $\operatorname{gr} d_w(B/C) = d_w(\operatorname{gr} B/R)$.*

Proof. Since $\operatorname{gr} b$ is a basis of $\operatorname{gr} B$ over R, $d_w(\operatorname{gr} B/R)$ is homogeneous of degree $N := 2\sum_{i=1}^{w} \deg(\operatorname{gr} b_i)$ (Lemma 2.6). Let σ be in S_w. By Lemma 4.9(2), $\deg \operatorname{tr}(b_i b_{\sigma(i)}) \leq \deg b_i + \deg b_{\sigma(i)}$, so $\deg \prod_{i=1}^{w} \operatorname{tr}(b_i b_{\sigma(i)}) \leq N$. Now we compute:

$$\begin{aligned}
\operatorname{gr} d_w(B/C) = \operatorname{gr}\,[\det(tr(b_i b_j))] &= \operatorname{gr}\,[\sum_{\sigma \in S_w} (-1)^{|\sigma|} \prod_{i=1}^{w} \operatorname{tr}(b_i b_{\sigma(i)})] \\
&\equiv_N \sum_{\sigma \in S_w} (-1)^{|\sigma|} \operatorname{gr}[\prod_{i=1}^{w} \operatorname{tr}(b_i b_{\sigma(i)})] \\
&\equiv_N \sum_{\sigma \in S_w} (-1)^{|\sigma|} \prod_{i=1}^{w} \operatorname{gr}[\operatorname{tr}(b_i b_{\sigma(i)})] \\
&\equiv_N \sum_{\sigma \in S_w} (-1)^{|\sigma|} \prod_{i=1}^{w} \operatorname{tr}(\operatorname{gr}[b_i b_{\sigma(i)}]) \\
&\equiv_N \sum_{\sigma \in S_w} (-1)^{|\sigma|} \prod_{i=1}^{w} \operatorname{tr}(\operatorname{gr}(b_i)\operatorname{gr}(b_{\sigma(i)})) \\
&\equiv_N d_w(\operatorname{gr} B/R).
\end{aligned}$$

The assertion follows. □

4.5. Locally nilpotent derivations

As in the previous subsection, let Λ be a totally ordered abelian semigroup and let B be a finitely generated Λ-filtered algebra. Let ∂ be a derivation of B. Let X be a set of generators of B as a k-algebra. Define the degree of ∂, denoted by $\deg \partial$, to be the maximal element of $\deg \partial(x) - \deg x$ for all $x \in X$ (to construct $\deg \partial(x) - \deg x$, one may have to pass to a totally ordered abelian group containing Λ). By the Leibniz rule, $\deg \partial(f) \leq \deg \partial + \deg f$ for all $f \in B$. Suppose $\deg \partial \in \Lambda$ exists, and define $\operatorname{gr} \partial$ by

$$(\operatorname{gr}\partial)(\operatorname{gr} f) = \begin{cases} 0 & \deg \partial(f) < \deg \partial + \deg f \\ \operatorname{gr}(\partial(f)) & \deg \partial(f) = \deg \partial + \deg f \end{cases}$$

for all $\operatorname{gr} f \in \operatorname{gr}_F B$. It is easy to see that this definition is independent of the choice of $f \in B$. The following lemma is not hard and the proof is omitted.

Lemma 4.11. *Let Λ be a totally ordered abelian semigroup and B be a finitely generated Λ-filtered algebra. If ∂ is a nonzero derivation, then $\operatorname{gr}\partial$ is a nonzero homogeneous derivation of degree $\deg\partial$. If ∂ is locally nilpotent, then so is $\operatorname{gr}\partial$.*

5. q-quantum Weyl algebras

Fix $q \in k^\times$ and let $A_q = k\langle x, y\rangle/(yx = qxy + 1)$. If $q = 1$, A_1 is the usual first Weyl algebra. In this section we assume that $q \neq 1$. When $q = -1$, the automorphism group of A_{-1} was studied in [6]. If $q \neq \pm 1$, it is well known that $\operatorname{Aut}(A_q) = k^\times$ [3]. The purpose of this section is not to give another proof this result, but to compute the discriminant of this algebra, in order to describe the automorphism group of other related algebras (such as the tensor product of A_q's).

Suppose q is a primitive nth root of unity for some $n \geq 2$. In keeping with the notation in previous sections, let $B = A_q$. We consider B as an $\mathbb{N}$-filtered algebra with $\deg x = 1$ and $\deg y = 0$. The following lemma is easy to check. We identify x and y with $\operatorname{gr} x$ and $\operatorname{gr} y$ in $\operatorname{gr} B$.

Lemma 5.1. *Retain the above notation and let q be a primitive nth root of unity for some $n \geq 2$.*

(1) *B is an $\mathbb{N}$-filtered algebra with $\deg x = 1$ and $\deg y = 0$ such that $\operatorname{gr} B = k_q[x, y]$ with $\deg x = 1$ and $\deg y = 0$.*
(2) *The center of B is $C := k[x^n, y^n]$. Let $R = \operatorname{gr} C$. Then R, which is the polynomial subalgebra $k[x^n, y^n]$ of $k_q[x, y]$, is the center of $\operatorname{gr} B$.*
(3) *There is a subset $b = \{x^i y^j \mid 0 \leq i, j \leq n\} \subset B$ such that B is a finitely generated free module over C with the basis b.*
(4) *$\operatorname{gr} B$ is a finitely generated free module over $R = \operatorname{gr} C$ with the basis $\operatorname{gr} b$.*
(5) *The condition (4.8.1) in Subsection 4.4 holds.*

Proposition 5.2. *Suppose q is a primitive nth root of unity with $n \geq 2$. Then*

$$d(A_q/C(A_q)) =_{k^\times} (x^n y^n)^{n(n-1)} + (\text{cwlt}).$$

As a consequence, $d(A_q/C(A_q))$ is dominating.

Proof. Retain the notation in Lemma 5.1, let $B = A_q$, $C = C(B)$ and $R = C(\operatorname{gr} B)$. By Proposition 2.8 (with $r = w = n^2$), $d_w(\operatorname{gr} B/R) = (x^n y^n)^{n(n-1)}$. By Proposition 4.10,

$\operatorname{gr} d_w(B/C) = (x^n y^n)^{n(n-1)}$. In particular, $d_w(B/C) \neq 0$. Write $d := d_w(B/C) = (x^n y^n)^{n(n-1)} + \sum_{i,j} a_{i,j} x^i y^j$ with $a_{ij} \in k$. Then the equation $d_w(\operatorname{gr} B/R) = (x^n y^n)^{n(n-1)}$ implies that

$$d = (y^{n^2(n-1)})x^{n^2(n-1)} + \sum_{i<n^2(n-1)} x^i (\sum_j a_{ij} y^j).$$

This means that if $a_{ij} \neq 0$, then $i < n^2(n-1)$. By symmetry (or using a different filtration of B), one sees that if $a_{ij} \neq 0$, then $j < n^2(n-1)$. Thus the assertion follows. $\square$

Based on computer calculations, we make the following conjecture.

Conjecture 5.3. *Suppose q is a primitive nth root of unity. Then*

$$d(A_q/C(A_q)) =_{k^\times} ((1-q)^n x^n y^n - 1)^{n(n-1)}.$$

This conjecture holds when $n = 2$: see [6, Example 1.7(1)].

[Added upon revision: this conjecture has now been proved: see [7].]

For the rest of this section we consider the tensor product of the q-quantum Weyl algebras A_q (the quantum version of the first Weyl algebra). Use the letter B for the tensor product $A_{q_1} \otimes \cdots \otimes A_{q_m}$. The following corollary follows immediately from Proposition 5.2 and [6, Theorem 5.5].

Corollary 5.4. *Let $B = A_{q_1} \otimes \cdots \otimes A_{q_m}$ and assume that each $1 \neq q_i$ is a root of unity. Then B is in* Af*, namely, $d(B/C(B))$ is dominating. As a consequence,* $\operatorname{Aut}(B)$ *is affine.*

From now on we do not assume that the parameters q_i are roots of unity. Here is the first part of Theorem 2.

Theorem 5.5. *Let $B = A_{q_1} \otimes \cdots \otimes A_{q_m}$ be defined as before. Assume that $q_i \neq 1$ for all $i = 1, \ldots, m$. Then every algebra automorphism of B is affine.*

Proof. Let Y be the subspace $\bigoplus_{i=1}^m (kx_i \oplus ky_i)$. Then Y is a generating space of B and B is a filtered algebra with standard filtration defined by $F_n B = (Y \oplus k)^n$ (and with $\deg x_i = \deg y_i = 1$ for all i). Clearly, $\operatorname{gr} B$ is a skew polynomial ring. So we have a monomial basis for the algebra B.

Proceed by contradiction and assume that there is an automorphism g of B which is not affine. Write $g(x_i)$, $g(y_i)$, $g^{-1}(x_i)$, $g^{-1}(y_i)$ as linear combinations of the monomial basis, and let K be the $\mathbb{Z}$-subalgebra of k generated by the collection of the nonzero coefficients $\{c_w\}_w$ of $g(x_i)$, $g(y_i)$, $g^{-1}(x_i)$, $g^{-1}(y_i)$, along with $\{c_w^{-1}\}_w$, $\{q_i^{\pm 1}\}_i$ and $\{(q_i - 1)^{-1}\}_i$. (If k is not a field, adjoin inverses as necessary.) Let S be the K-subalgebra of B generated by $\{x_i\}_{i=1}^m \bigcup \{y_i\}_{i=1}^m$. By the definition of K, both g and g^{-1} are well-defined as algebra homomorphisms of S. Since $g \circ g^{-1}$ and $g^{-1} \circ g$ are the identity when restricted to the K-subalgebra $S \subset B$, g is an automorphism of S with inverse g^{-1}. Since the relations of B (and of S) are of the form

$$y_i x_i = q_i x_i y_i + 1, \quad [x_i, x_j] = [x_i, y_j] = [y_i, y_j] = 0 \tag{5.5.1}$$

for all $i \neq j$, one can check that $\operatorname{gr} S$ is a skew polynomial algebra with base ring K (or S is an iterated Ore extension starting with K). In fact, it is free over K. Hence the hypotheses of Lemma 4.6 hold.

Now consider a finite quotient field $F = K/\mathfrak{m}$. Then the image of q_i, denoted by $\bar{q}_i$, is not 1 in F. Since S is an iterated Ore extension, $S \otimes_K F$ is also an iterated Ore extension with the relation (5.5.1) with q_i being replaced by $\bar{q}_i$. Therefore $S \otimes_K F$ is isomorphic to the product of quantum Weyl algebras $A_{\bar{q}_i}$ over the field F, where $\bar{q}_i \neq 1$. Since F is a finite field, $\bar{q}_i$ is a root of unity. By Corollary 5.4 $\operatorname{Aut}(S \otimes_K F) = \operatorname{Aut}_{\mathrm{af}}(S \otimes_K F)$. By Lemma 4.6(1), $\operatorname{Aut}(S) = \operatorname{Aut}_{\mathrm{af}}(S)$, which contradicts the fact that $g|_S$ is not affine. The assertion follows. $\square$

To prove the rest of Theorem 2 (and Theorem 5.7 below), we need the following lemma.

Lemma 5.6. *Let $B = A_{q_1} \otimes \cdots \otimes A_{q_m}$ with $q_i \neq 1$ for all i. Let Y be the subspace $\bigoplus_{i=1}^m (kx_i \oplus ky_i)$. Let g be a (necessarily affine) automorphism of B.*

(1) *$g(Y) = Y$.*
(2) *For each i, either $g(x_i) = b_i x_{i'}$ and $g(y_i) = f_i y_{i'}$ for some i', or $g(x_i) = c_i y_{i'}$ and $g(y_i) = e_i x_{i'}$ for some i'.*
(3) *If k is a field, then $\operatorname{Aut}_{\mathrm{af}}(B)$ is an algebraic group that fits into the exact sequence*

$$1 \to (k^\times)^m \to \operatorname{Aut}_{\mathrm{af}}(B) \to S \to 1,$$

where S is the finite group generated by all automorphisms g of the form $g(x_i) = x_{i'}$ and $g(y_i) = y_{i'}$ for some i', or $g(x_i) = y_{i'}$ and $g(y_i) = x_{i'}$ for some i'.
(4) *If $q_i \neq q_j^{-1}$ for all i, j, then there is a permutation $\sigma \in S_m$ and $b_i \in k^\times$ such that $g(x_i) = b_i x_{\sigma(i)}$ and $g(y_i) = b_i^{-1} y_{\sigma(i)}$ for all i. Further $q_i = q_{\sigma(i)}$ for all i.*
(5) *If $q_i \neq \pm 1$ and $q_i \neq q_j^{\pm 1}$ for all $i \neq j$, then $\operatorname{Aut}_{\mathrm{af}}(B) = (k^\times)^m$.*
(6) *If $q_i = q \neq \pm 1$ for all i, then $\operatorname{Aut}_{\mathrm{af}}(B) = S_m \ltimes (k^\times)^m$.*

Proof. (1) Write

$$g(x_i) = a_i + \sum_{s=1}^m b_{is} x_s + \sum_{t=1}^m c_{it} y_t = a_i + X_i,$$

$$g(y_i) = d_i + \sum_{s=1}^m e_{is} x_s + \sum_{t=1}^m f_{it} y_t = d_i + Y_i,$$

where $a_i, b_{is}, c_{it}, d_i, e_{is}, f_{it} \in k$. Applying g to the relation $1 = y_i x_i - q x_i y_i$ (where we write $q = q_i$), we have

$$\begin{aligned} 1 &= g(y_i)g(x_i) - qg(x_i)g(y_i) \\ &= Y_i X_i + a_i Y_i + d_i X_i + a_i d_i - q(X_i Y_i + a_i Y_i + d_i X_i + a_i d_i) \\ &= Y_i X_i - q X_i Y_i + (1-q)[a_i Y_i + d_i X_i + a_i d_i]. \end{aligned}$$

By using the relations of B, the degree 1 part of the above equation is

$$0 = (1-q)[a_i Y_i + d_i X_i].$$

Since $q \neq 1$, $a_i Y_i + d_i X_i = 0$. If a_i or d_i is nonzero, then X_i and Y_i are linearly dependent, which contradicts the fact that $\{1, x_i, y_i\}$ is linearly independent. Therefore $a_i = d_i = 0$ for all i. The assertion follows.

(2) We keep the notation from part (1), and we know that $a_i = d_i = 0$ for all i. Note that the x_s's commute and the y_t's commute. Then

$$\begin{aligned} 1 &= g(y_i)g(x_i) - q_i g(x_i)g(y_i) = Y_i X_i - q_i X_i Y_i \\ &= \left(\sum_{s=1}^m e_{is}x_s + \sum_{t=1}^m f_{it}y_t\right)\left(\sum_{s=1}^m b_{is}x_s + \sum_{t=1}^m c_{it}y_t\right) \\ &\quad - q_i \left(\sum_{s=1}^m b_{is}x_s + \sum_{t=1}^m c_{it}y_t\right)\left(\sum_{s=1}^m e_{is}x_s + \sum_{t=1}^m f_{it}y_t\right) \\ &= (1-q_i)\left[\left(\sum_{s=1}^m e_{is}x_s\right)\left(\sum_{s=1}^m b_{is}x_s\right) + \left(\sum_{t=1}^m f_{it}y_t\right)\left(\sum_{t=1}^m c_{it}y_t\right)\right] \\ &\quad + \left(\sum_{s=1}^m e_{is}x_s\right)\left(\sum_{t=1}^m c_{it}y_t\right) + \left(\sum_{t=1}^m f_{it}y_t\right)\left(\sum_{s=1}^m b_{is}x_s\right) \\ &\quad - q_i\left(\sum_{s=1}^m b_{is}x_s\right)\left(\sum_{t=1}^m f_{it}y_t\right) - q_i\left(\sum_{t=1}^m c_{it}y_t\right)\left(\sum_{s=1}^m e_{is}x_s\right). \end{aligned}$$

By using the monomial basis of B, one sees that

$$\left(\sum_{s=1}^m e_{is}x_s\right)\left(\sum_{s=1}^m b_{is}x_s\right) = \left(\sum_{t=1}^m f_{it}y_t\right)\left(\sum_{t=1}^m c_{it}y_t\right) = 0.$$

Since B is a domain, we have either

$$\sum_{s=1}^m e_{is}x_s = 0 = \sum_{t=1}^m c_{it}y_t$$

or

$$\sum_{s=1}^{m} b_{is}x_s = 0 = \sum_{t=1}^{m} f_{it}y_t.$$

In the first case, the equation becomes

$$1 = \sum_{s\neq t} b_{is}f_{it}(1-q_i)x_sy_t + \sum_{s} b_{is}f_{is}[(q_s-q_i)x_sy_s+1].$$

This implies that $b_{is}f_{it} = 0$ for all $s \neq t$. As a consequence, b_{is} is zero except for one s and f_{it} is zero except for one t. The assertion follows. The argument for the second case is similar.

(3) This follows from part (2).

(4) Suppose that $g(x_i) = c_iy_{i'}$ and $g(y_i) = e_ix_{i'}$ for some i'. Applying g to $1 = y_ix_i - q_ix_iy_i$ we have

$$1 = c_ie_i(x_{i'}y_{i'} - q_iy_{i'}x_{i'}) = c_ie_i[(1-q_iq_{i'})x_{i'}y_{i'} - q_i].$$

which implies that $q_i = q_{i'}^{-1}$, a contradiction. By part (2), we have that for each i, $g(x_i) = b_ix_{i'}$ and $g(y_i) = f_iy_{i'}$. Further, by the relation, one has that $q_i = q_{i'}$ and $f_i = b_i^{-1}$. The assignment $i \mapsto i'$ defines the required permutation σ. Finally it is easy to check that $q_i = q_{\sigma(i)}$ for all i.

(5,6) Follows from part (4). □

Theorem 5.7. *Let $B = A_{q_1} \otimes \cdots \otimes A_{q_m}$. Assume that $q_i \neq 1$ for all i. Then*

(1) *Every automorphism of B is affine. As a consequence, the following hold.*
   (a) *If $q_i \neq \pm 1$ and $q_i \neq q_j^{\pm 1}$ for all $i \neq j$, then $\mathrm{Aut}(B) = (k^\times)^m$.*
   (b) *If $q_i = q \neq \pm 1$ for all i, then $\mathrm{Aut}(B) = S_m \rtimes (k^\times)^m$.*
(2) *The automorphism group of $B[t]$ is triangular.*
(3) *If k is a field, then $\mathrm{Aut}(B)$ is an algebraic group that fits into the exact sequence*

$$1 \to (k^\times)^m \to \mathrm{Aut}(B) \to S \to 1$$

   for some finite group S.
(4) *If $\mathbb{Z} \subset k$, $\mathrm{LNDer}(B) = \{0\}$.*

Proof. The main assertion in part (1) is Theorem 5.5. Parts (a,b) follow from the main assertion and Lemma 5.6(5,6).

The proof of part (2) is similar to the proof of Theorem 5.5 and omitted.

(3) This is a consequence of part (1) and Lemma 5.6(3).

(4) By localizing k, we may assume that $\mathbb{Q} \subseteq k$. Then this is a consequence of part (2) and [6, Lemma 3.3(2)]. □

Theorem 2 follows from Theorem 5.7.

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